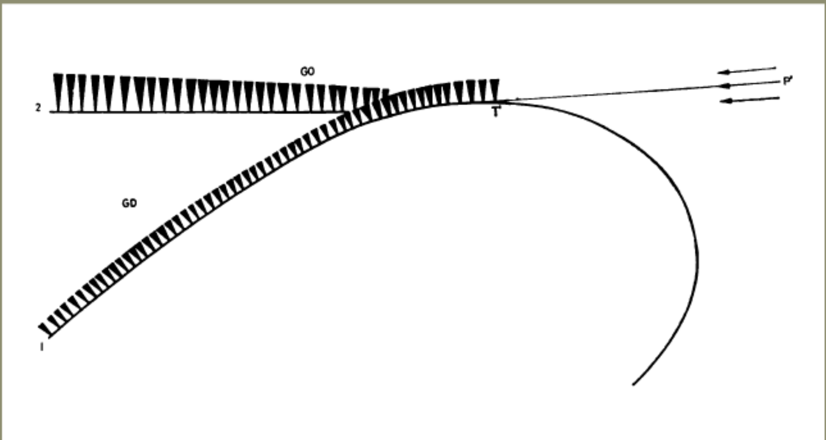
V. A. FOCK

Electromagnetic Diffraction and Propagation Problems



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VOLUME 1

ELECTROMAGNETIC DIFFRACTION AND PROPAGATION PROBLEMS

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by

V. A. FOCK

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ERRATA

Page 7	Eq. (1.8)	*For* $\xi = l\sqrt[3]{\left(\frac{\lambda}{\pi}R_0^2\right)} = l\,d$	*Read* $\xi = l:\sqrt[3]{\left(\frac{\lambda}{\pi}R_0^2\right)} = l:d$
16	Line 21	ξ_u	ξ_u
Page 52	Fq. (9.07)	*For* $2iuH_u = \sqrt{(uv)}\left\{2\frac{\partial Q^*}{\partial u\partial v} - \frac{\partial^2 Q}{\partial v} - \frac{1}{4}Q + \frac{1}{2}\frac{\partial P}{\partial u} + \frac{1}{2}\frac{\partial P}{\partial v}\right\}\sin\varphi,$	*Read* $2iuH_u = \sqrt{(uv)}\left\{2\frac{\partial Q^*}{\partial v} - \frac{\partial^2 Q}{\partial u\partial v} - \frac{1}{4}Q + \frac{1}{2}\frac{\partial P}{\partial u} + \frac{1}{2}\frac{\partial P}{\partial v}\right\}\sin\varphi,$
Page 74	Line 9	*For* $\Gamma(s-it)/2$	*Read* $\Gamma\left(\frac{s-it}{2}\right)$
Page 79	Eq. (2.30)	*For* $\log\tan\frac{\delta_1}{2} = \log \mathrm{tg}\frac{\delta}{2} - q,$	*Read* $\log\tan\frac{\delta_1}{2} = \log\tan\frac{\delta}{2} - q,$
Page 142	Line 3	*For* the product of the functions two	*Read* the product of the two functions
Page 193	Line 9	*For* $\ln(k_2) > 0$	*Read* $Im(k_2) > 0$
Page 317	Eq. (4.16)	*For* $g_1(\zeta) = D_{i\nu-\frac{1}{2}}\zeta e^{-i\frac{\pi}{4}}$	*Read* $g_1(\zeta) = D_{i\nu-\frac{1}{2}}\left(\zeta e^{-i\frac{\pi}{4}}\right)$
Page 317	Eq. (4.17)	*For* $g_2(\zeta) = D_{-i\nu-\frac{1}{2}}\zeta e^{i\frac{\pi}{4}}$	*Read* $g_2(\zeta) = D_{-i\nu-\frac{1}{2}}\left(\zeta e^{i\frac{\pi}{4}}\right)$
Page 332	Line 7 from bottom	*For* (5.08)	*Read* (5.08))
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INTRODUCTION

NEW physical concepts are formed not only in the process of generalization of a physical theory, but also in the inverse way: they can arise from approximate methods applied to a more comprehensive physical theory. Thus the concept of a ray (or, more generally, geometrical optics) can be deduced from a wave theory of light in the limiting case of vanishing wavelength, if the field near the boundary between light and shadow is not considered in detail. The next step is the consideration of deviations from geometrical optics, or, in other words, of diffraction (which is again a new concept); diffraction phenomena appear most strikingly just near the boundary between light and shadow.

In the original treatment of diffraction phenomena the specific properties of the material of the diffracting body were not taken into account, the body being considered as perfectly black (absorbing). The Fresnel reflection formulae which take these properties into account were not used in the analysis of diffraction phenomena. The basis of diffraction theory in its original form was the Huygens–Fresnel principle; on that basis the field near the boundary between light and shadow was described with the help of Fresnel's integrals and the field near the caustic with the help of Airy's integrals.

With the advent of electromagnetic theory a reformulation of diffraction theory became necessary. The firm theoretical basis for the description of diffraction phenomena are the Maxwell equations with the corresponding boundary and radiation conditions. The asymptotic diffraction theory can be defined as an approximation to the Maxwell theory valid in the limiting case of small wavelengths and large radii of curvature of the diffracting bodies.

The asymptotic theory has introduced its own principles. We have the principle of the local field in the penumbra region on the surface of a convex body, with its generalization for the region outside the body, and the approximate form of the boundary conditions valid for good conductors (Leontovich); Leontovich's conditions are of impedance type and follow from the local character of the field in the surface layer of the conducting body. These principles permitted us to obtain explicit expressions for

the field valid on and near the surface of a conducting body of arbitrary shape. (Our formulae, though perfectly general, were originally obtained by considering the diffraction from a paraboloid of revolution, but they have been recently deduced anew by Cullen [28] by solving the integral equation established in Chapter 2 directly).

A further principle is that of different scales for horizontal and vertical distances; this enables us to replace the full wave equation by the parabolic equation (Leontovich). This latter is like the Schrödinger equation of quantum mechanics (or like the diffusion equation with an imaginary diffusion coefficient) and permits us to introduce the notion of transverse diffusion (Malyughinetz).

In the first part of this book the asymptotic diffraction theory is developed. This is done partly on the basis of exact solutions of the Maxwell equations (which solutions are then transformed into an approximate form that allows physical interpretation and is suitable for numerical calculations), and partly by means of a direct application of the principles stated above, especially of the parabolic wave equation. In some cases the two methods are used independently; this enables us to make a comparison between them and confirms the principles underlying the "asymptotic" method. In most cases the field is considered only near the surface of the diffracting body, but in Chapter 17 the field at arbitrary distances is also considered.

The subject of the second part of this book is radio wave propagation. The Chapters 10, 11 and 12 of Part II in which formulae for the homogeneous atmosphere are developed belong to the domain of pure diffraction and are thus a direct continuation of Part I. In the Chapters 13 to 17 the case of an inhomogeneous (stratified) atmosphere is considered, with refractive index depending only on the height.

It is worth while noticing that the case of a stratified atmosphere offers another example of new physical concepts arising from approximations: for instance the notions of the modified refractive index (combining the effects of refraction and of the curvature of the earth), of an atmospheric waveguide with trapped waves or (as a complementary concept) with waves reflected from the boundaries of the waveguide, and the notion of the horizon determining the propagation range in the case of superrefraction.

The aim of this book is mainly to expound the general theory as developed by the author rather than to give numerical results. Nevertheless, in cases when interpretation of the theoretical results is not simple, numerical results in the form of graphs and tables are also given. An Appendix is added, containing four-figure tables of the Airy functions, together with

a description of the main properties and of some mathematical applications of these functions.

Since this book is a collection of papers in their original form (with a few corrections only), some repetitions were unavoidable. But it is hoped that the book clearly shows the development of the author's ideas in the theory of diffraction and propagation of electromagnetic waves.

V. FOCK

PART ONE

Asymptotic Theory of Diffraction

CHAPTER 1

NEW METHODS IN DIFFRACTION THEORY†

THE general problem of the theory of diffraction of electromagnetic waves consists in finding a solution of Maxwell's equations, having prescribed singularities (field sources) and satisfying prescribed boundary conditions and conditions at infinity.

The solution of this problem presents mathematical difficulties, which arise chiefly from the necessity of taking into account the geometrical shape of the obstacles on which the wave is falling. The problem is somewhat simplified if only monochromatic waves of given frequency are considered, but the difficulties are still so great that the problem has not yet been solved, except in cases when the obstacle is of a particularly simple form. The best known of these are the cases of a perfectly reflecting half-plane or a wedge, the cases of a sphere and a circular cylinder.

The cases of an elliptic and a parabolic cylinder have also been considered, and the field of a plane wave incident on a perfectly reflecting paraboloid of revolution (oblique incidence) has recently been obtained by the author. In the few cases enumerated a rigorous solution of the problem in the form of an infinite series of integrals has been obtained.

The aim of a theory is to give a picture reproducing all the qualitative and quantitative features of the phenomenon considered. This aim is not attained until the solution obtained is of a sufficiently simple form. If the rigorous solution has a complicated analytical form, it constitutes only the first step; a second step must be made—the derivation of formulae suitable for numerical calculations.

This second step may be as difficult as the first one. To give an example, we may mention that the problem of diffraction of electromagnetic waves around a sphere was solved rigorously some 40 years ago (Mie). This problem includes that of the propagation of radio-waves along the surface of the earth. Owing to the slow convergence of the series involved, the

† Fock, 1948.

general solution could, however, not be applied to the latter problem until 1918, when a transformation of the original series into another rapidly converging series was found (Watson). But the improved form of the solution was still unsatisfactory in some respects, being very complicated and applicable only in the region of the geometrical shadow (far beyond the line of horizon). A far more satisfactory form of the solution, applicable in all cases of practical importance, has been recently found by the author [10]. Thus, the way from the rigorous theoretical solution to the approximate practical one took about 40 years of research.

A possible procedure then, is to find first a rigorous solution of a diffraction problem and then to transform it into another form suitable for numerical calculations – this straightforward method is, however, of very limited application. It can only be applied to the few problems admitting a rigorous solution in form of series of integrals.

In other cases (especially when the diffracting obstacle is of arbitrary shape) attempts have been made to reduce the problem to integral equations. These attempts have proved successful from the theoretical point of view; but with the exception of a paper by the author [2], no use has been made of the integral equations for the practical solution of the problem, the general theory of integral equations being quite useless for purposes of numerical calculation.

An approximate method, sufficiently general and leading to sufficiently simple formulae is thus urgently needed. In the following we shall outline the principal ideas of such a method, proposed and developed by the author.

Every approximate method is based on the smallness of some parameters involved in the problem. We have to consider which of the parameters of our problem may be regarded as small.

We are usually concerned with the propagation of waves in air, i.e. in a medium with properties widely different from those of the scattering bodies (obstacles). The electrical properties of these bodies are characterized by means of the complex dielectric permittivity

$$\eta = \varepsilon + i\frac{4\pi\sigma}{\omega} \tag{1.1}$$

(ε denotes as usual the dielectric constant, σ – the conductivity of the medium, ω – the frequency). Now it is essential that in most cases $|\eta| \gg 1$. Thus we may choose as one of the small parameters of the problem the inverse value of $|\eta|$ or the quantity $1/\sqrt{|\eta|}$.

Next, the wavelength λ *in vacuo* is usually very much smaller than the

radii of curvature of the scattering bodies. We thus have another small parameter — the quotient λ/R, where R is the radius of curvature of the obstacle. It is convenient to take instead the quantity

$$\frac{1}{m} = \sqrt[3]{\left(\frac{\lambda}{\pi R}\right)} \tag{1.2}$$

In addition to the two small parameters defined above, there may be others, depending on the position of the point of observation. For instance, in the problem of the propagation of radio-waves along the earth's surface the angle of inclination of the ray to the horizon may be regarded as small.

Let us consider the consequences of the fact that the parameters $1/\sqrt{|\eta|}$ and $1/m$ are small. In the limiting case $|\eta| = \infty$ (perfect conductor) a great simplification arises from the fact that the field is known beforehand inside the conductor (this field being equal to zero). We can confine ourselves to the space outside the conductor by prescribing proper boundary conditions to the field in air (the tangential components of the electrical vector should vanish at the surface). A similar situation arises if $\sqrt{|\eta|}$ is very large. The field inside the body is in this case very small except in a thin surface layer (skin–effect), and the influence of this layer may be accounted for by stating boundary conditions for the external field. These are of the form

$$\frac{4\pi}{c} j_x = \sqrt{\eta} \cdot (E_x - n_x E_n) = n_y H_z - n_z H_y \qquad \text{etc.} \tag{1.3}$$

where (j_x, j_y, j_z) is the surface current density vector, (n_x, n_y, n_z) the unit vector of the normal to the surface, E_n the normal component of the electric field, the meaning of the other symbols being evident. These conditions, first stated by Leontovich [11, 21] in a somewhat different form, apply if $|\eta| \gg 1$ and if $kR\sqrt{|\eta|} \gg 1$ ($k = 2\pi/\lambda$). The latter inequality signifies that the thickness of the skin layer should be small as compared with the radius of curvature of the obstacle. Conditions (1.3) may be easily generalized for arbitrary values of the magnetic permeability μ.

Consequently the smallness of $1/\sqrt{|\eta|}$ permits us to confine our attention to the field outside and on the body, which constitutes an important simplification of the problem.

We now proceed to examine the influence of the smallness of the wavelength. As is well known, in the limiting case of small wavelengths the laws of geometrical optics become valid. Particularly, the boundary of the shadow on the surface of the body becomes sharp and well defined. On one side of the boundary — in the illuminated region — the field obeys very

nearly Fresnel's laws of reflection, and on the dark side the field rapidly decreases to zero.

The approximation given by geometrical optics is, however, not sufficient for our purposes. The point of interest is the diffraction phenomenon in its strict sense, i.e. the bending of the ray around the obstacle. This phenomenon cannot be treated by means of geometrical optics, and to give a theory of this phenomenon a more accurate solution of the field equations is required.

The author succeeded in finding this solution by means of a new principle which may be called the "Principle of the Local Field in the Penumbra Region".

This principle consists of the following: – The transition from light to shadow on the surface of the body takes place in a narrow strip along the boundary of the geometrical shadow. The width of this strip is of the order

$$d = \sqrt[3]{\left(\frac{\lambda}{\pi} R_0^2\right)} \tag{1.4}$$

where R_0 is the radius of curvature of the surface of the body in the plane of incidence. It may be proved that, neglecting small quantities of order $\sqrt[3]{\left(\frac{\lambda}{\pi R_0}\right)}$, the field in this strip has a local character; it depends only on the value of the field of the incident wave in the neighbourhood of the point considered, on the geometrical shape of the body near that point and on the electrical properties of the material of the body. The field near a given point on the strip does not depend on its values at distant points and can be calculated separately.

To establish the principle of the local field and to derive explicit formulae for this field, two different methods have been used.

One of these see (Chapter 2) applies to the case of a perfect conductor and gives the values of the field on its surface. We start with the integral equation for the surface current density j. This is of the form

$$j = 2j^{ex} + \frac{1}{2\pi} \int \left\{ \frac{n \times [j' \times (r-r')]}{R^3} f \right\}_{surf} dS' \tag{1.5}$$

where

$$f = (1-ikR)\, e^{ikR} \tag{1.6}$$

The vector j^{ex} (external current density) is defined by the expression (1.3), where H is replaced by H^{ex}, the magnetic vector of the external field; r is the radius vector of the point of observation, r' that of the point of integration; $R = |r-r'|$ is the length of the chord between r and r'; n

is the unit vector along the normal at r. A qualitative study of the integral equation permits us to establish the principle of the local field. This principle having been established, we have to find a solution of the diffraction problem for a convex body of a particular shape and to derive approximate formulae for the field on its surface. By virtue of the principle of the local field, these formulae hold for any other convex body having at the point considered the same values of the principal radii of curvature. (The particular body must of course be sufficiently general to possess points with any prescribed values of principal radii of curvature; actually a paraboloid of revolution has been used.) Proceeding in this way we arrive at a general formula for the surface value of the tangential component of the magnetic field or, which amounts to the same, for the surface current density vector. This formula is of the form

$$j = j^{ex} G(\xi, 0) \tag{1.7}$$

where the argument ξ of G denotes the quantity

$$\xi = l : \sqrt[3]{\left(\frac{\lambda}{\pi} R_0^2\right)} = l : d \tag{1.8}$$

l being the distance from the boundary of the geometrical shadow, measured along the ray (i.e., along the line of intersection of the plane of incidence with the surface of the body) and taken positive in the direction of the shadow and negative in the opposite direction. The function $G(\xi, 0)$ is defined by the integral

$$G(\xi, 0) = e^{i\frac{\xi^3}{3}} \cdot \frac{1}{\sqrt{\pi}} \int_C \frac{e^{i\xi t}}{w'(t)} dt \tag{1.9}$$

where C is a contour in the complex t-plane running from infinity to zero along the line arc $t = 2\pi/3$ and from zero to infinity along the positive real axis.

The function $w(t)$ may be called the complex Airy function; it is defined by the differential equation

$$w''(t) = tw(t) \tag{1.10}$$

and by the asymptotic behaviour for large negative values of t

$$w(t) = e^{i\frac{\pi}{4}} \cdot (-t)^{-\frac{1}{4}} \exp\left[i\frac{2}{3}(-t)^{\frac{3}{2}}\right] \tag{1.11}$$

The function $G(\xi, 0)$ tends to the limit $G = 2$ for large negative values of ξ while its modulus decreases exponentially for large positive values of ξ.

Formula (1.7) reproduces thus the gradual decrease of the field amplitude when passing from light to shadow.

The same results may be obtained by another method [5] which allows us to generalize them in two respects. Firstly, the body need not be a perfect conductor, but may have a finite conductivity, provided the boundary conditions (1.3) are applicable. Secondly, the field is obtained not only on the surface of the body, but also near the surface (at distances that are small compared with the radii of curvature). The method consists in simplifying Maxwell's equations and the boundary conditions by neglecting quantities of the order of the square of the small parameters $1/\sqrt{|\eta|}$ and $1/m$. The wave equation for the amplitude is thereby replaced by a parabolic equation of Schrödinger's type. The simplified equations are valid in a limited region near a point on the penumbra strip.

The solution of these equations may be performed by means of the separation of variables and yields the field in the region considered and especially in the penumbra strip on the body. Introducing the complex quantity

$$q = \frac{im}{\sqrt{\eta}} = \frac{i}{\sqrt{\eta}} \sqrt[3]{\left(\frac{\pi R_0}{\lambda}\right)} \tag{1.12}$$

(the modulus $|q|$ is thus the quotient of the two small parameters), we may write instead of (1.7)

$$j = j^{ex}_{\omega}\, G(\xi, q) \tag{1.13}$$

where

$$G(\xi, q) = e^{i\frac{\xi^3}{3}} \frac{1}{\sqrt{\pi}} \int_C \frac{e^{i\xi t}\, dt}{w'(t) - qw(t)} \tag{1.14}$$

the contour C being the same as in (1.9). These formulae give the distribution of currents on the penumbra strip and generalize our previous formulae (1.7) and (1.9). The formulae for the field near the surface are more complicated and will not be written here.

It is to be noted that in the outer portion of the strip, where the illuminated region begins, approximate expressions can be derived from our formulae that coincide with expressions for the field obtained by superposing the incident and the reflected wave and using Fresnel's coefficients of reflection. On the other hand, in the opposite portion of the strip the field is practically zero. Thus our formulae constitute the missing link joining the two regions where the laws of geometrical optics may be applied. Together with Fresnel's formulae they allow us to calculate the field near and on the whole surface of the diffracting body.

In some problems this is all that is required. In the problem of propagation of waves around the earth's surface, for instance, we are only concerned with the field at heights not exceeding ten kilometres – a quantity that is small compared with the earth's radius (6380 km). In this instance our formulae, if modified so as to include the case when the source is near or on the surface, give the required solution.

In other problems, however, the field at large distances from the scattering body is needed. In spite of the fact that our formulae are valid only in the region near the surface, they provide a means of calculating the field at large distances also. Indeed, the field of the scattered wave is generated by the currents induced on the surface (in the skin layer) by the incident wave. These currents are given by our formulae. Thus, by applying well-known theorems on the vector potential due to a given current distribution, we may, in principle, calculate the field for arbitrary distances from the reflecting body.

The principle of the local field in the penumbra region thus provides a basis for the approximate solution of the diffraction problem in the general case of a convex body of arbitrary shape.

CHAPTER 2

THE DISTRIBUTION OF CURRENTS INDUCED BY A PLANE WAVE ON THE SURFACE OF A CONDUCTOR†

Abstract — The problem considered is to find the distribution of currents, induced on the surface of a perfectly conducting body by an incident plane wave. The body is supposed to be convex and to have a continuously varying curvature. The wavelength λ of the incident wave is supposed to be small compared with the dimensions of the body and with the radii of curvature of its surface. It is shown that the current distribution in the vicinity of the geometrical shadow is expressible in terms of a universal function $G(\xi)$ (the same for all bodies), depending on the argument $\xi = l/d$, where l is the distance from the boundary of the geometrical shadow, measured in the plane of incidence, and d is the width of the penumbra region $\left(d = \sqrt[3]{\left(\frac{\lambda}{\pi} R_0^2\right)}\right.$ where R_0 is the radius of curvature of the surface of the body in the plane of incidence). For the function $G(\xi)$ an analytical expression is derived and tables are computed.

Let us consider a perfectly conducting body on the surface of which a plane electromagnetic wave is falling. The surface of the conductor is supposed to be convex, with a continuously varying curvature. The incident wave induces on the conductor electrical currents, which in their turn become a source of the scattered wave. If the current distribution on the conductor is determined, then the calculation of the field of the scattered wave may be performed by applying the well-known formulae for the vector potential. Hence the essential step in solving the problem of diffraction of a plane wave by a perfect conductor is to find the currents induced on its surface.

The present paper is a preliminary report on our work concerning the approximate solution of this problem.

1. Let us denote by j the surface current density on the conductor. The vector j is defined for every point of the surface and is directed along

† Fock, 1945.

the tangent to the surface. It is completely determined by its two tangential components, the third component (normal to the surface) being equal to zero.

It can be shown that the vector j satisfies the following integral equation:

$$j = 2j^{ex} + \frac{1}{2\pi} \int \left\{ \frac{n \times [j' \times (r - r')]}{R^3} f \right\}_{\text{surf}} dS' \tag{1.01}$$

with

$$f = (1 - ikR)\, e^{ikR} \tag{1.02}$$

In this equation R is the length of the chord joining the two points of the surface, the fixed point $r(x, y, z)$ for which the integral is evaluated, and the variable point $r'(x', y', z')$, whose coordinates are functions of the integration variables. n is a unit vector along the normal to the surface at the point r, dS' is the surface element at r' and k is the absolute value of the wave vector.

The quantity j^{ex} is the external current density defined by the formula

$$j^{ex} = \frac{c}{4\pi} [n \times H^{ex}] \tag{1.03}$$

where H^{ex} is the value of the magnetic field of the incident wave on the surface (external field).

If the dependence of the external field upon the coordinates is given by the factor

$$e^{ik(\alpha x + \beta y + \gamma z)} \tag{1.04}$$

then the current density may be sought in the form of a product of a similar factor with a slowly varying function of coordinates. The integral (1.01) after dividing by (1.04) takes the form

$$I = \int e^{ik[R + \alpha(x' - x) + \beta(y' - y) + \gamma(z' - z)]} \varphi \, dS' \tag{1.05}$$

where φ is a slowly varying function. If the wavelength is sufficiently small compared with the dimensions of the body, the value of the integral will be approximately

$$I = \frac{2\pi i}{k} \frac{R}{\cos\theta} \varphi \tag{1.06}$$

where the point x', y', z' is connected with the point x, y, z as is shown in Figs. 1 and 2, and θ is the angle of incidence of the ray.

The analytical connection between the points x', y', z' and x, y, z is given by the following formulae. Let n' denote the unit vector along the

normal at the point x', y', z' and let

$$\left.\begin{aligned} \alpha+2n'_x\cos\theta &= \alpha^* \\ \beta+2n'_y\cos\theta &= \beta^* \\ \gamma+2n'_z\cos\theta &= \gamma^* \end{aligned}\right\} \tag{1.07}$$

where

$$\cos\theta = -(\alpha n'_x+\beta n'_y+\gamma n'_z) \tag{1.08}$$

The quantities α^*, β^*, γ^* are the direction cosines of the ray reflected at the point x', y', z'

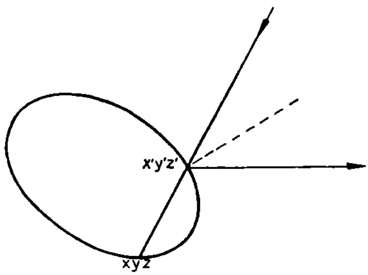

FIG. 1.

With these notations, we have either:

$$\frac{x-x'}{R} = \alpha; \qquad \frac{y-y'}{R} = \beta; \qquad \frac{z-z'}{R} = \gamma \tag{1.09}$$

or

$$\frac{x-x'}{R} = \alpha^*; \qquad \frac{y-y'}{R} = \beta^*; \qquad \frac{z-z'}{R} = \gamma^* \tag{1.10}$$

the formulae (1.09) being valid, if the point x', y', z' is situated on the illuminated part of the surface (Fig. 1), while (1.10) are valid if this point is situated on the shadow part of the surface. In the latter case the reflected ray is fictitious.

With the same degree of approximation as in formula (1.06) the integral equation (1.01) allows the following solution:

$$\begin{aligned} j &= 2j^{ex} \text{ on the illuminated part,} \\ j &= 0 \text{ on the shadow part.} \end{aligned} \tag{1.11}$$

Near the boundary of the geometrical shadow (where $\cos\theta \simeq 0$), formula (1.06) ceases to be valid and expression (1.11) does not give a gradual transition from light to shadow.

2. In order to obtain an expression for the currents valid in the transition region also, it is necessary to use a more exact solution. It is rather difficult to derive it directly from the integral equation†, but we have succeeded in obtaining it in an indirect way, on the basis of the following considerations.

First of all, it is seen from Figs. 1 and 2 that if the point x, y, z lies near the geometrical boundary of the shadow, the point x', y', z' lies also near this boundary and near the point x, y, z. Therefore, the value of the integral (1.01) is determined by the values of the integrand in the neighbourhood of the point for which the integral is evaluated. Thus, in the region of the

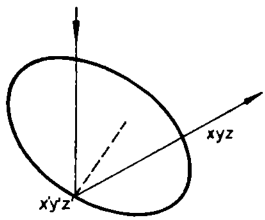

FIG. 2.

penumbra (near the geometrical boundary of the shadow) the field has a local character. Secondly, the investigation of the integral equation (carried out under the assumption that the chord can be replaced by its projection on the tangent plane) shows that the width of the penumbra region is of the order of

$$d = \sqrt[3]{\left(\frac{\lambda}{\pi} R_0^2\right)} \tag{2.01}$$

where R_0 is the radius of curvature of the surface of the body in the plane of incidence. But in a region of width d and in a certain more extended region the nucleus of the integral equation depends essentially only on the curvature of the surface in the neighbourhood of a given point (i.e. on the second but not on the higher derivatives of the surface equation with respect to coordinates).

Hence it follows, that all bodies with a smoothly varying curvature have the same current distribution in the penumbra region, if the curvature and the incident wave are the same near the point under consideration.

The results stated permit us to infer that, if we solve the problem for

† This was done by S.A. Cullen in 1958 [28].

a particular case, we can obtain universal formulae for the field on the surface of a perfect conductor. These formulae apply immediately to the region of the penumbra, but the field may be considered as known everywhere on the surface, since for the illuminated region and for the remote shaded region the expressions (1.11) are valid.

3. The derivation of these universal formulae is too complicated to be given in any detailed form in a short paper†. We confine ourselves to some indication of the method, and to the statement of the result, which may be done in quite a simple way.

The considerations developed above show that we can take as a starting point for the derivation of the general formulae an exact solution of the problem of diffraction of a plane wave by some convex body with a smoothly varying curvature. The surface of the body must, of course, be sufficiently general, i.e. must possess points with given values of the principal radii of curvature.

There are two cases in which exact solutions of the problem are known, namely, the case of a sphere and the case of a circular cylinder (in the last case the incidence of the wave is supposed to be normal). These bodies are, however, not sufficiently general: for a sphere the two radii of curvature are equal, and for a cylinder one of the radii is infinite. The simplest of the bodies having arbitrary values of the curvature radii are the ellipsoid and the paraboloid of revolution. For these bodies only the general form of the solution of the scalar wave equation is known; the complete solution of Maxwell's equation for the given physical problem appears to be unknown.

In our work we have obtained the required solution for the paraboloid of revolution (particularly the values of the tangential components of the magnetic field on its surface) and have used this solution to derive the approximate formulae.

Let the equation of the paraboloid have the form

$$x^2+y^2-2az-a^2 = 0 \tag{3.01}$$

The components of the field of the incident wave are

$$\left.\begin{aligned} E_x &= E_0 \cos \delta\, e^{i\Omega}, & H_x &= 0 \\ E_y &= 0, & H_y &= E_0\, e^{i\Omega} \\ E_z &= -E_0 \sin \delta\, e^{i\Omega}; & H_z &= 0 \end{aligned}\right\} \tag{3.02}$$

where

$$\Omega = k(x \sin \delta + z \cos \delta) \tag{3.03}$$

† See Chapters 3 and 4.

If the parabolic coordinates:

$$u = k\,(r+z)$$
$$v = k\,(r-z) \tag{3.04}$$
$$\varphi = \operatorname{arc\,tan}\frac{y}{x}$$

with

$$r = \sqrt{(x^2+y^2+z^2)} \tag{3.05}$$

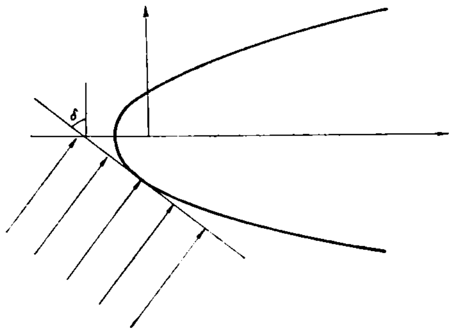

FIG. 3.

are introduced, the equation of the paraboloid becomes

$$v = v_0 = ka \tag{3.06}$$

For the generalized (covariant) tangential components of the external magnetic field we have the expressions

$$2iuH_u^{\mathrm{ex}} + H_\varphi^{\mathrm{ex}} = \frac{E_0}{k}\sqrt{(uv)}\,\mathrm{e}^{i\Omega+i\varphi} \tag{3.07}$$

$$-2iuH_u^{\mathrm{ex}} + H_\varphi^{\mathrm{ex}} = \frac{E_0}{k}\sqrt{(uv)}\,\mathrm{e}^{i\Omega-i\varphi} \tag{3.08}$$

In the new coordinates the expression for Ω has the form

$$\Omega = \frac{1}{2}(u-v)\cos\delta + \sqrt{(uv)}\sin\delta\cos\varphi \tag{3.09}$$

For the same components of the total field, expressions in the form of Fourier series with respect to the angle φ are obtained. The coefficients of $\sin s\varphi$ and $\cos s\varphi$ in these series are definite integrals with respect to the

parameter t, involving some complicated functions of u, v, δ, s, t. These series and integrals can be transformed into double integrals of the form

$$2iuH_u+H_\varphi = \frac{E_0\sqrt{(uv)}}{2\pi k \sin\delta}\int\int g(s, t)\exp\left(-is\varphi+it\lg\tan\frac{\delta}{2}\right)\mathrm{d}s\,\mathrm{d}t \quad (3.10)$$

where the function $g(s, t)$ is defined in the following way. Let $\zeta(v, s, t)$ be an integral of the differential equation

$$v\frac{\mathrm{d}^2\zeta}{\mathrm{d}v^2}+\frac{\mathrm{d}\zeta}{\mathrm{d}v}+\left(\frac{v}{4}-\frac{s^2}{4v}+\frac{t}{2}\right)\zeta = 0 \quad (3.11)$$

having at $v\to\infty$ an asymptotic form

$$\zeta(v, s, t) = \mathrm{e}^{-\frac{\pi}{4}t-i\frac{s+1}{4}\pi}\, v^{-\frac{1}{2}+\frac{it}{2}}\,\mathrm{e}^{i\frac{v}{2}}\, F_{20}\left(\frac{1-s-it}{2};\ \frac{1+s-it}{2};\ -\frac{i}{v}\right) \quad (3.12)$$

where F_{20} is an asymptotic series of the form

$$F_{20}\left(\alpha, \beta, \frac{1}{x}\right) = 1+\frac{\alpha\beta}{1}\frac{1}{x}+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2}\frac{1}{x^2}+\dots \quad (3.13)$$

We put

$$N(s, t) = \frac{i}{\pi}\mathrm{e}^{\pi t-i\frac{\pi}{2}s}\frac{\Gamma\left(\frac{s-it}{2}\right)\Gamma\left(\frac{-s-it}{2}\right)}{\zeta^2(v, s, -t-i)+\frac{1}{4}(s^2+t^2)\zeta^2(v, s, -t+i)} \quad (3.14)$$

where v is given by equation (3.06).

Then

$$g(s, t) = \mathrm{e}^{is\frac{\pi}{2}}\zeta(u, s+1, t)\zeta(v, s-1; t)(s-it)N(s, t) \quad (3.15)$$

With $g(s, t)$ having this value, the expression (3.10) is valid, if $-\pi/2<\varphi<\pi/2$. In the cases $\pi/2<\varphi<3\pi/2$ and $-3\pi/2<\varphi<-\pi/2$ we have to take for $g(s, t)$ a somewhat different expression, which we shall not write down here. The integration in (3.10) with respect to the variable t is to be made along the real axis from $-\infty$ to $+\infty$ and with respect to s along the imaginary axis from $-i\infty$ to $+i\infty$. The value of $-2i_uH_u+H_\varphi$ is obtained from (3.10), if we replace φ by $-\varphi$.

The double integral can be evaluated approximately assuming that the value of $v=ka$ is very large. Let us introduce the quantity

$$\xi = \frac{\sqrt{uv}\sin\delta\cos\varphi-v\cos\delta}{[2v(u+v)]^{\frac{1}{3}}(\sin\delta)^{\frac{2}{3}}} \quad (3.16)$$

It is easy to verify that on the geometrical boundary of the shadow we have $\xi=0$; but in general ξ will be large, of the order of $v^{1/3}$. Therefore, when evaluating the integrals we shall consider v to be very large and ξ to be arbitrary (in general, finite). It can be shown, that under these assumptions the following approximate expressions for the integrals are valid with a relative error of the order of $v^{-1/3}$:

$$2iuH_u+H_\varphi = \frac{E_0}{k}\sqrt{(uv)}\,\mathrm{e}^{i\Omega+i\varphi}\,G(\xi) \tag{3.17}$$

$$-2iuH_u+H_\varphi = \frac{E_0}{k}\sqrt{(uv)}\,\mathrm{e}^{i\Omega-i\varphi}\,G(\xi). \tag{3.18}$$

where

$$G(\xi) = \mathrm{e}^{i\frac{\xi^3}{3}}\frac{1}{\sqrt{\pi}}\int_{\Gamma_1}\frac{\mathrm{e}^{i\xi\tau}\,\mathrm{d}\tau}{w'(\tau)} \tag{3.19}$$

the symbol Γ_1 denoting a contour running from infinity to the origin along the path arc $z=2\pi/3$ and from the origin to infinity along the path arc $z=0$ (the positive real axis).

The function $w(\tau)$ whose derivative is involved in the integrand satisfies the differential equation

$$w''(\tau) = \tau\, w(\tau) \tag{3.20}$$

and can be written in the form of an integral

$$w(\tau) = \frac{1}{\sqrt{\pi}}\int_{\Gamma_2}\mathrm{e}^{\tau z-\frac{1}{3}z^3}\,\mathrm{d}z \tag{3.21}$$

where the contour denoted by Γ_2 runs from infinity to the origin along the path arc $z=-2\pi/3$ and from the origin to infinity along the positive real axis.

Comparison of (3.17) and (3.18) with (3.07) and (3.08) gives

$$H_{\mathrm{tg}} = H_{\mathrm{tg}}^{ex}G(\xi) \tag{3.22}$$

Thus the tangential components of the total magnetic field are equal to the tangential components of the external field multiplied by a certain complex function of a single variable ξ. A similar relation exists between the total and the "external" current density, namely

$$j = j^{ex}G(\xi) \tag{3.23}$$

Let us examine the geometrical meaning of the variable ξ in more detail. Consider the section of the paraboloid surface in the plane of incidence

passing through the given point (Fig.4). We denote by l the distance of the given point from the geometrical boundary of the shadow, considered positive in the direction of the shadow and negative in the direction of the light. The distance l is measured in the plane of incidence. Let R_0 be the radius of curvature of the surface section and $k = 2\pi/\lambda$, the absolute value of the wave vector.

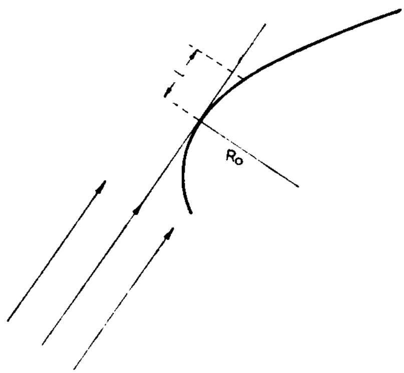

FIG. 4.

Then the quantity

$$\xi = \sqrt[3]{\left(\frac{k}{2R_0^2}\right)} l = \frac{l}{d} \tag{3.24}$$

(where d is the width (2.01) of the penumbra region) is easily seen to coincide with the quantity (3.16) defined for a paraboloid of revolution. Since we know beforehand that formulae (3.22) and (3.23) are quite general, we conclude that they are valid for all bodies with a given curvature, if ξ is given by (3.24).

These formulae give the transition from the shadow to the light.

For large positive values of ξ the function $G(\xi)$ is approximately equal to

$$G(\xi) = c\, \mathrm{e}^{i\left(\frac{\xi^3}{3}+a\xi\right)} \mathrm{e}^{-b\xi} \tag{3.25}$$

where a, b, c are known numbers, namely

$$a = 0{\cdot}5094; \quad b = 0{\cdot}8823; \quad c = 1{\cdot}8325 \tag{3.26}$$

Owing to the factor $\mathrm{e}^{-b\xi}$ the function $G(\xi)$ decreases rapidly. This corresponds to the decrease of the amplitude in the shadow region.

For large negative values of ξ the function $G(\xi)$ admits an asymptotic expansion of the form

$$G(\xi) = 2+\frac{i}{2\xi^3}+\ldots \tag{3.27}$$

and tends to a limit which is equal to 2. This limiting value corresponds to formulae (1.11) for the illuminated region. The discontinuous function (1.11) is thus replaced in our more exact solution by the continuous function (3.23). This enables us to calculate the distribution of currents on the surface of a conducting body with sufficient accuracy.

In the Appendix, tables of the function G defined by (3.19) are given and also of the function g related to G by the equation

$$G(x) = e^{i\frac{x^3}{3}} g(x) \tag{3.28}$$

and expressible in the form of the integral

$$g(x) = \frac{1}{\sqrt{\pi}}\int_{\Gamma_1} \frac{e^{ixt}}{w'(t)}\,dt \tag{3.29}$$

The function $G(x)$ is tabulated for values of x from $x=-4{\cdot}5$ to $x=1$ in intervals of 0·1, and the function $g(x)$ is tabulated for a range of values of x from $x=-1$ to $x=4{\cdot}5$ with the same intervals. For values of x less than $x=-4{\cdot}5$ expression (3.27) may be used, and for values of x greater than $x=4{\cdot}5$ formula (3.25) becomes applicable.

Appendix

Table of the function $G(x) = e^{i\frac{x^3}{3}} g(x)$

x	Re G	Im G	$\lvert G\rvert$	arc G
−4·5	1·9998	−0·0055	1·9998	− 9′ 30″
−4·4	1·9997	−0·0059	1·9997	−10′ 10″
−4·3	1·9997	−0·0063	1·9997	−10′ 50″
−4·2	1·9996	−0·0067	1·9997	−11′ 40″
−4·1	1·9996	−0·0073	1·9996	−12′ 30″
−4·0	1·9995	−0·0078	1·9995	−13′ 20″
−3·9	1·9994	−0·0084	1·9995	−14′ 30″
−3·8	1·9994	−0·0090	1·9994	−15′ 30″
−3·7	1·9992	−0·0098	1·9993	−16′ 50″
−3·6	1·9991	−0·0106	1·9991	−18′ 10″

(Contd.)

x	Re G	Im G	$\lvert G \rvert$	arc G
−3·5	1·9990	−0·0115	1·9990	− 19′ 40′
−3·4	1·999	−0·012	1·999	− 21′
−3·3	1·999	−0·014	1·999	− 23′
−3·2	1·998	−0·015	1·998	− 26′
−3·1	1·998	−0·016	1·998	− 28′
−3·0	1·998	−0·018	1·988	− 31′
−2·9	1·997	−0·020	1·997	− 34′
−2·8	1·996	−0·022	1·996	− 37′
−2·7	1·996	−0·024	1·996	− 41′
−2·6	1·995	−0·026	1·995	− 46′
−2·5	1·993	−0·029	1·994	− 51′
−2·4	1·992	−0·033	1·992	− 56′
−2·3	1·990	−0·036	1·990	− 1°03′
−2·2	1·988	−0·040	1·988	− 1°10′
−2·1	1·985	−0·045	1·985	− 1°18′
−2·0	1·981	−0·050	1·982	− 1°27′
−1·9	1·977	−0·056	1·977	− 1°37′
−1·8	1·971	−0·062	1·972	− 1°47′
−1·7	1·965	−0·068	1·966	− 1°58′
−1·6	1·956	−0·075	1·958	− 2°11′
−1·5	1·946	−0·082	1·948	− 2°25′
−1·4	1·933	−0·090	1·936	− 2°40′
−1·3	1·919	−0·098	1·921	− 2°55′
−1·2	1·901	−0·105	1·904	− 3°10′
−1·1	1·880	−0·113	1·884	− 3°27′
−1·0	1·857	−0·119	1·861	− 3°40′
−0·9	1·829	−0·123	1·833	− 3°51′
−0·8	1·798	−0·126	1·802	− 4°00′
−0·7	1·762	−0·126	1·766	− 4°05′
−0·6	1·722	−0·122	1·726	− 4°03′
−0·5	1·678	−0·115	1·682	− 3°54′
−0·4	1·630	−0·103	1·633	− 3°36′
−0·3	1·578	−0·086	1·580	− 3°06′
−0·2	1·522	−0·063	1·523	− 2°22′
−0·1	1·462	−0·034	1·463	− 1°21′
0	1·399	0	1·399	0°00′
0·1	1·333	0·040	1·334	1°44′
0·2	1·263	0·086	1·266	3°55′
0·3	1·189	0·137	1·197	6°35′
0·4	1·111	0·193	1·128	9°51′
0·5	1·029	0·252	1·059	13°45′
0·6	0·941	0·312	0·991	18°22′
0·7	0·846	0·373	0·924	23°47′
0·8	0·744	0·432	0·860	30°08′
0·9	0·634	0·484	0·798	37°22′
1·0	0·515	0·529	0·738	45°44′

TABLE OF THE FUNCTION $g(x) = e^{-i\frac{x^3}{3}} G(x)$

x	Re g	Im g	$\lvert g \rvert$	arc g
−1·0	1·794	0·495	1·861	15°26′
−0·9	1·805	0·320	1·833	10°04′
−0·8	1·793	0·181	1·802	5°47′
−0·7	1·765	0·076	1·766	2°28′
−0·6	1·726	0·002	1·726	0°04′
−0·5	1·681	−0·045	1·682	− 1°31′
−0·4	1·632	−0·068	1·633	− 2°23′
−0·3	1·578	−0·071	1·580	− 2°35′
−0·2	1·522	−0·059	1·523	− 2°13′
−0·1	1·462	−0·034	1·463	− 1°20′
0	1·399	0	1·399	0°00′
0·1	1·333	0·040	1·334	1°43′
0·2	1·263	0·083	1·266	3°45′
0·3	1·190	0·127	1·197	6°04′
0·4	1·115	0·169	1·128	8°37′
0·5	1·038	0·209	1·059	11°21′
0·6	0·961	0·244	0·991	14°14′
0·7	0·883	0·274	0·924	17°14′
0·8	0·806	0·299	0·860	20°19′
0·9	0·732	0·317	0·798	23°27′
1·0	0·660	0·331	0·738	26°38′
1·1	0·591	0·339	0·682	29°50′
1·2	0·527	0·343	0·628	33°02′
1·3	0·467	0·342	0·578	36°13′
1·4	0·411	0·338	0·532	39°25′
1·5	0·360	0·330	0·488	42°34′
1·6	0·313	0·320	0·448	45°42′
1·7	0·270	0·309	0·410	48°48′
1·8	0·232	2·960	0·376	51°53′
1·9	0·197	0·281	0·343	54°56′
2·0	0·167	0·267	0·315	57°59′
2·1	0·140	0·252	0·289	61°00′
2·2	0·116	0·237	0·264	64°00′
2·3	0·095	0·222	0·242	66°58′
2·4	0·076	0·208	0·221	69°56′
2·5	0·0596	0·1936	0·2025	72°54′
2·6	0·0453	0·1797	0·1853	75°51′
2·7	0·0330	0·1664	0·1696	78°47′
2·8	0·0224	0·1536	0·1552	81°43′
2·9	0·0133	0·1414	0·1421	84°39′
3·0	−0·0055	0·1299	0·1300	87°34′
3·1	−0·0010	0·1190	0·1190	90°30′
3·2	−0·0065	0·1088	0·1089	93°25′
3·3	−0·0110	0·0991	0·0997	96°20′
3·4	−0·0147	0·0901	0·0913	99°15′

(Contd.)

x	Re g	Im g	$\lvert g \rvert$	arc g
3·5	−0·0176	0·0817	0·0836	102°10′
3·6	−0·0199	0·0739	0·0765	105°05′
3·7	−0·0216	0·0666	0·0700	108°00′
3·8	−0·0229	0·0599	0·0641	110°55′
3·9	−0·0237	0·0537	0·0587	113°50′
4·0	−0·0242	0·0480	0·0537	116°45′
4·1	−0·0244	0·0428	0·0492	119°40′
4·2	−0·0243	0·0380	0·0451	122°35′
4·3	−0·0240	0·0336	0·0413	125°30′
4·4	−0·0235	0·0296	0·0378	128°25′
4·5	0·0228	0·0260	0·0346	131°20′

CHAPTER 3

THEORY OF DIFFRACTION BY A PARABOLOID OF REVOLUTION†

INTRODUCTION

The present paper consists of two parts. In the first part the theory of parabolic functions is developed (Sections 1–3) and some expansions in series of these functions are given (Sections 4 and 5). In the following sections of the first part the theory of the solution of Maxwell's equations in parabolic coordinates is developed (Sections 6 and 8). The most important result of the first part is the introduction of the parabolic potentials P and Q, permitting formulation of the boundary conditions without recourse to finite difference equations; this procedure considerably simplifies all problems connected with a paraboloid of revolution. The expressions for the field are simplified, if four interrelated auxiliary functions connected with the parabolic potentials P, Q are introduced (Section 7).

Thus, the first part contains the mathematical formalism necessary for the solution of diffraction problems connected with a paraboloid of revolution.

In the second part the radiation of a dipole located at the focus of an absolutely reflecting paraboloid of revolution is considered. The primary field of the dipole is expressed in terms of the parabolic potentials of the general theory (Section 9). For the reflected wave the potentials are first expressed in terms of integrals (Section 10) and then in terms of series of two types, with different regions of convergence (Section 11). The auxiliary functions are also expressed in terms of series. In the last two sections (Section 12 and Section 13) the field in the wave zone is investigated. Explicit expressions are given for this field, corresponding to the geometrical optics approximation (including first order corrections). In particular, the dependence of the field amplitude on the distance from the axis is obtained.

† Fock, 1957.

The most important application of the theory of parabolic potentials is perhaps the solution of the diffraction problem for a plane wave incident at an arbitrary angle on the outer surface of a paraboloid of revolution. This problem is of particular interest in connection with the principle of the local field in the penumbra region, put forward in our previous papers. A brief summary of the solution of this problem is given in Chapter 2. A more detailed derivation of the solution is given in Chapter 4, where the general case of the polarization of the incident plane wave is also considered.

General Theory

1. *Parabolic Coordinates*

Let x, y, z denote rectangular coordinates. We write the equation of the paraboloid of revolution in the form

$$x^2+y^2-2az-a^2 = 0 \tag{1.01}$$

Let k be the absolute value of the wave vector

$$k = \frac{2\pi}{\lambda_0} \tag{1.02}$$

(where λ_0 denotes the wavelength). We take as independent variables the following quantities: firstly, the angle φ between a fixed plane and the plane going through the z-axis and the given point (the fixed plane may be chosen as the $x0z$-plane); this angle will be the same as in usual cylindrical coordinates; secondly, the parabolic coordinates u, v, connected with the rectangular ones according to the formulae

$$u = k(R+z); \qquad v = k(R-z), \tag{1.03}$$

$$r = \frac{1}{k}\sqrt{(uv)}; \qquad z = \frac{1}{2k}(u-v); \qquad R = \frac{1}{2k}(u+v), \tag{1.04}$$

where

$$r = \sqrt{(x^2+y^2)}, \qquad R = \sqrt{(x^2+y^2+z^2)}. \tag{1.05}$$

The rectangular coordinates are expressed in terms of the parabolic coordinates by means of the formulae

$$x = \frac{1}{k}\sqrt{(uv)}\cos\varphi; \qquad y = \frac{1}{k}\sqrt{(uv)}\sin\varphi; \qquad z = \frac{1}{2k}(u-v). \tag{1.06}$$

The coordinate surfaces u=constant, v=constant represent two fam-

ilies of mutually-orthogonal paraboloids of revolution. The equation of the given paraboloid (1.01) is of the form

$$v = v_0 \qquad \text{where} \qquad v_0 = ka \tag{1.07}$$

as it is easy to verify by direct substitution of (1.06) into (1.01). To the outer region (exterior with respect to the given paraboloid) correspond values $v > v_0$; to the inner (interior) region correspond values $v < v_0$. The coordinate u varies within the limits $0 \leqslant u < \infty$.

The square of the line element in parabolic coordinates is of the form

$$\mathrm{d}s^2 = \frac{1}{k^2}\left(\frac{u+v}{4u}\,\mathrm{d}u^2 + \frac{u+v}{4v}\,\mathrm{d}v^2 + uv\,\mathrm{d}\varphi^2\right). \tag{1.08}$$

As is always the case when curvilinear orthogonal coordinates are used, one has to distinguish between the projections of a vector on the given coordinate direction and the covariant components of a vector (the latter are quantities with the same transformation law as for the partial derivatives of a scalar function with respect to the coordinate variables). We shall denote the projections of the physical vector by parentheses, for example, (E_u), (E_v), (E_φ), keeping the notation E_u, E_v, E_φ for the covariant components. Then we have

$$\left.\begin{aligned} (E_u) &= \frac{k\sqrt{u}}{\sqrt{(u+v)}}\,E_u \\ (E_v) &= \frac{k\sqrt{v}}{\sqrt{(u+v)}}\,E_v \\ (E_\varphi) &= \frac{k}{2\sqrt{(uv)}}\,E_\varphi \end{aligned}\right\} \tag{1.09}$$

and similarly for other vectors.

2. *Parabolic Functions with a Continuous Parameter*

The Laplace operator in parabolic coordinates has the form

$$\Delta\psi = \frac{4k^2}{u+v}\left\{\frac{\partial}{\partial u}\left(u\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial\psi}{\partial v}\right) + \frac{u+v}{4uv}\frac{\partial^2\psi}{\partial\varphi^2}\right\}. \tag{2.01}$$

Therefore, the wave equation

$$\Delta\psi + k^2\psi = 0 \tag{2.02}$$

can be written as

$$\frac{\partial}{\partial u}\left(u\frac{\partial\psi}{\partial u}\right) + \frac{\partial}{\partial v}\left(v\frac{\partial\psi}{\partial v}\right) + \frac{1}{4}\left(\frac{1}{u} + \frac{1}{v}\right)\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{4}(u+v)\psi = 0. \tag{2.03}$$

Putting

$$\psi = U(u)V(v)\,e^{is\varphi} \tag{2.04}$$

and introducing this expression into (2.03), we see that the variables in equation (2.03) separate, and we obtain for the functions U and V the equations

$$u\frac{d^2U}{du^2}+\frac{dU}{du}+\left(\frac{u}{4}-\frac{s^2}{4u}+\frac{t}{2}\right)U = 0 \tag{2.05}$$

$$v\frac{d^2V}{dv^2}+\frac{dV}{dv}+\left(\frac{v}{4}-\frac{s^2}{4v}-\frac{t}{2}\right)V = 0 \tag{2.06}$$

where t is the separation parameter.

In order that the solution (2.04) should be a single-valued function in space, the quantity s must be an integer (in the equations (2.05) and (2.06) one can obviously take s to be positive, $s > 0$). The separation parameter t, however, can assume arbitrary real or complex values. But it is often useful to introduce for the solutions integral representations in which s plays the part of an integration variable. To include these cases, we shall consider U and V as analytical functions of the variable s.

The equations obtained for the functions U and V are of the same form (only the sign of t is different). Their solutions are well known.

We first consider that solution of equation (2.05) which is finite at $u=0$. This solution can be written as

$$U = \xi(u, s, t),$$

where

$$\xi(u, s, t) = \frac{u^{\frac{s}{2}}\,e^{-i\frac{u}{2}}}{\Gamma\left(\frac{s+1+it}{2}\right)\Gamma\left(\frac{s+1-it}{2}\right)}\times$$

$$\times\int_0^1 e^{iuz}\,z^{\frac{s-1+it}{2}}(1-z)^{\frac{s-1-it}{2}}\,dz. \tag{2.07}$$

Expanding the integral in a power series, we have also

$$\xi(u, s, t) = e^{-i\frac{u}{2}}\sum_{k=0}^{\infty} i^k\frac{u^{\frac{s}{2}+k}}{k!}\,\frac{\Gamma\left(\frac{s+1+it}{2}+k\right)}{\Gamma\left(\frac{s+1+it}{2}\right)\Gamma(s+k+1)} \tag{2.08}$$

or

$$\xi(u, s, t) = e^{-i\frac{u}{2}} u^{\frac{s}{2}} \frac{1}{\Gamma(s+1)} F\left(\frac{s+1+it}{2}, \quad s+1, \quad iu\right), \tag{2.09}$$

where $F(\alpha, \gamma, x)$ is the confluent hypergeometric series

$$F(\alpha, \gamma, x) = 1 + \frac{\alpha}{\gamma} \cdot \frac{x}{1} + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)} \frac{x^2}{1\cdot 2} + \dots \tag{2.10}$$

Interchanging the quantities z and $1-z$ in the integral (2.07) one can easily see that, for real values of u, s, t the function $\xi(u, s, t)$ will be real. It is seen from (2.09) that $\xi(u, s, t)$ is an integral transcendental function of t and of s.

Let us now consider the second solution of equation (2.03), namely that whose asymptotic expression for large u contains the factor $e^{i\frac{u}{2}}$. This solution is obtained from (2.07) if we integrate with respect to z not from 0 to 1, but from $1+i\infty$ to 1. The constant factor before the integral can be, of course, chosen in another way than in (2.07). Transforming the integral by the substitution

$$1-z = \frac{x}{u} e^{-i\frac{\pi}{2}}$$

and choosing the constant factor in an appropriate way, we can take as the second solution of (2.05) the expression

$$\zeta_1(u, s, t) = e^{-\frac{\pi}{4}t - i\frac{s+1}{4}\pi} \cdot u^{-\frac{1}{2}+\frac{it}{2}} \cdot e^{i\frac{u}{2}} \cdot F_{20}, \tag{2.11}$$

where

$$F_{20} = \frac{1}{\Gamma\left(\frac{s+1-it}{2}\right)} \cdot \int_0^\infty e^{-x} x^{\frac{s-1-it}{2}} \left(1 + i\frac{x}{u}\right)^{\frac{s-1+it}{2}} dx. \tag{2.12}$$

Expression (2.12) admits an asymptotic expansion in powers of $1/u$, valid as $u \to \infty$. If we put

$$F_{20}\left(\alpha, \beta, \frac{1}{x}\right) = 1 + \frac{\alpha\beta}{1} \cdot \frac{1}{x} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2} \cdot \frac{1}{x^2} + \dots, \tag{2.13}$$

then expression (2.12) will be equal to

$$F_{20} = F_{20}\left(\frac{1-s-it}{2}, \quad \frac{1+s-it}{2}, \quad -\frac{i}{u}\right). \tag{2.14}$$

Unlike $\xi(u, s, t)$, the function $\zeta(u, s, t)$ will be complex for real values of u, s, t as well. But this function satisfies the equation (2.05) which has real coefficients. If follows that the function

$$\zeta_2(u, s, t) = e^{-\frac{\pi}{4}t+i\frac{s+1}{4}\pi} u^{-\frac{1}{2}-\frac{it}{2}} e^{-i\frac{u}{2}} \times$$
$$\times F_{20}\left(\frac{1-s+it}{2}, \quad \frac{1+s+it}{2}, \quad \frac{i}{u}\right), \tag{2.15}$$

obtained from $\zeta_1(u, s, t)$ by interchanging i and $-i$ will satisfy the same equation. For all values of u, s, t the functions ζ_1 and ζ_2 will be linearly-independent solutions of the differential equation (2.05). Like ξ, the functions ζ_1 and ζ_2 are integral transcendental functions of the parameters s and t. When the sign of s is changed, we will have

$$\left.\begin{aligned} \zeta_1(u, -s, t) &= e^{i\frac{s\pi}{2}} \zeta_1(u, s, t) \\ \zeta_2(u, -s, t) &= e^{-i\frac{s\pi}{2}} \zeta_2(u, s, t). \end{aligned}\right\} \tag{2.16}$$

The functions are connected by the following relations. We have

$$\xi(u, s, t) = \frac{\zeta_1(u, s, t)}{\Gamma\left(\frac{s+1+it}{2}\right)} + \frac{\zeta_2(u, s, t)}{\Gamma\left(\frac{s+1-it}{2}\right)}. \tag{2.17}$$

With help of (2.17) and of a similar equation with the sign of s changed, and using the relations (2.16), we can express

$$\zeta_1(u, s, t) \quad \text{and} \quad \zeta_2(u, s, t)$$

in terms of

$$\xi(u, s, t) \quad \text{and} \quad \xi(u, -s, t).$$

We obtain

$$\zeta_1(u, s, t) = \frac{i\pi}{\sin s\pi} e^{-i\frac{\pi}{2}} \left\{ \frac{e^{-i\frac{s\pi}{2}}}{\Gamma\left(\frac{-s+1-it}{2}\right)} \xi(u, s, t) - \right.$$
$$\left. - \frac{1}{\Gamma\left(\frac{s+1-it}{2}\right)} \xi(u, -s, t) \right\}. \tag{2.18}$$

The function ζ_2 is obtained from (2.18) by changing i into $-i$. If the series (2.08) is inserted for ξ in this equation, an expansion of $\zeta_1(u, s, t)$ in ascen-

ding powers of u is obtained. When s tends to an integer, the right-hand side of (2.18) tends to a finite limit; the limiting series for ζ_1 involves logarithmic terms.

Between the functions $\xi(u, s, t)$ with parameters s, differing by ± 1 and with parameters t, differing by $\pm 2i$ various recurrence relations exist, of which we note the following:

$$2u \frac{\partial}{\partial u} \xi(u, s, t) + \xi(u, s, t)$$
$$= \frac{s+1+it}{2} \xi(u, s, t-2i) + \frac{s+1-it}{2} \xi(u, s, t+2i), \tag{2.19}$$

$$i(u+t)\xi(u, s, t)$$
$$= \frac{s+1+it}{2} \xi(u, s, t-2i) - \frac{s+1-it}{2} \xi(u, s, t+2i), \tag{2.20}$$

$$\frac{s+it}{2} \xi(u, s, t-i) + \frac{s-it}{2} \xi(u, s, t+i) = \sqrt{u} \cdot \xi(u, s-1, t), \tag{2.21}$$

$$\xi(u, s, t-i) - \xi(u, s, t+i) = i\sqrt{u} \cdot \xi(u, s+1, t). \tag{2.22}$$

It is sometimes convenient to introduce in the calculations instead of $\xi(u, s, t)$ the function

$$\psi(u, s, t) = \Gamma\left(\frac{s+1+it}{2}\right) \xi(u, s, t), \tag{2.23}$$

which is, in contrast to $\xi(u, s, t)$, not an integral transcendental but a meromorphic function of s and t. The recurrence relations for $\psi(u, s, t)$ are easily obtained from (2.19)–(2.22). But these relations are the same as those satisfied by the function $\zeta_1(u, s, t)$. Consequently we have

$$2u \frac{\partial}{\partial u} \zeta_1(u, s, t) + \zeta_1(u, s, t)$$
$$= \zeta_1(u, s, t-2i) + \frac{1}{4}(s+1-it)(s-1+it)\zeta_1(u, s, t+2i); \tag{2.24}$$

$$i(u+t)\zeta_1(u, s, t)$$
$$= \zeta_1(u, s, t-2i) - \frac{1}{4}(s+1-it)(s-1+it)\zeta_1(u, s, t+2i); \tag{2.25}$$

$$\zeta_1(u, s, t-i) + \frac{1}{2}(s-it)\zeta_1(u, s, t+i)$$
$$= \sqrt{u} \cdot \zeta_1(u, s-1, t); \tag{2.26}$$

$$\zeta_1(u, s, t-i)-\frac{1}{2}(s+it)\zeta_1(u, s, t+i)$$
$$= i\sqrt{u}\cdot\zeta_1(u, s+1, t). \tag{2.27}$$

The recurrence relations for the function $\zeta_2(u, s, t)$ follow from (2.24) – (2.27) by changing i into $-i$.

For the estimate of various integrals and series involving parabolic functions, asymptotic expressions for these functions are required which are valid for large values of $|t|$. The asymptotic expression for $\xi(u, s, t)$ is of the form

$$\xi(u, s, t) \sim \left(\frac{t}{2}\right)^{-\frac{s}{2}} J_s\left(\sqrt{(2ut)}\right), \tag{2.28}$$

where J_s is the Bessel function. This expression is valid for finite and for small values of u (including $u=0$) if $|t|\gg 1$. For the functions ζ_1 and ζ_2 we will have in the upper half-plane of the complex variable t

$$\zeta_1(u, s, t) = \frac{\pi e^{-t\frac{\pi}{2}} e^{i\frac{s\pi}{4}}}{\Gamma\left(\frac{1-it}{2}\right)} H_s^{(1)}\left(\sqrt{(2ut)}\right), \tag{2.29}$$

$$\zeta_2(u, s, t) = \frac{1}{2} e^{-i\frac{s\pi}{4}} \Gamma\left(\frac{1-it}{2}\right)\times$$
$$\times\left\{H_s^{(2)}\left(\sqrt{(2ut)}\right) - e^{is\pi - t\pi} H^{(1)}\left(\sqrt{(2ut)}\right)\right\} \tag{2.30}$$

and in the lower half-plane

$$\zeta_1(u, s, t) = \frac{1}{2} e^{i\frac{s\pi}{4}} \Gamma\left(\frac{1+it}{2}\right)\times$$
$$\times\left\{H_s^{(1)}\left(\sqrt{(2ut)}\right) - e^{-is\pi - t\pi} H_s^{(2)}\left(\sqrt{(2ut)}\right)\right\}, \tag{2.31}$$

$$\zeta_2(u, s, t) = \frac{\pi e^{-t\frac{\pi}{2}} e^{-i\frac{s\pi}{4}}}{\Gamma\left(\frac{1+it}{2}\right)} H_s^{(2)}\left(\sqrt{(2ut)}\right). \tag{2.32}$$

The symbols $H_s^{(1)}$ and $H_s^{(2)}$ denote the first and the second Hankel functions. The functions $\Gamma\left(\frac{1-it}{2}\right)$ and $\Gamma\left(\frac{1+it}{2}\right)$ in the above formulae can be replaced by their asymptotic expressions.

In problems with axial symmetry and in those which are reducible to the former, functions with $s=0$ are often used. For brevity, they will be denoted by $\xi(u, t)$ (instead of $\xi(u, 0, t)$) and by $\zeta_1(u, t)$ (instead of $\zeta_1(u, 0, t)$) and similarly for other functions.

3. *Parabolic Functions with an Integral Index*

When solving diffraction problems connected with a paraboloid of revolution, it is expedient to use for some purposes integral representations and for other purposes series expansions of the solution. Integral representations involve functions ξ, ζ_1, ζ_2, studied in the preceding section. Series expansions proceed in functions with an integral index n, and it is useful to introduce a special notation for them, although they are expressible in terms of the functions studied above.

We put

$$\xi_{ns}(u) = \xi(u, s, -i(2n+s+1)) = \frac{\zeta_1(u, s, -i(2n+s+1))}{\Gamma(n+s+1)} \tag{3.01}$$

$$\eta_{ns}(u) = (-1)^n n!\, \zeta_2(u, s, -i(2n+s+1)). \tag{3.02}$$

Both functions, ξ_{ns} and η_{ns}, are solutions of the differential equation

$$u\frac{d^2\xi_{ns}}{du^2}+\frac{d\xi_{ns}}{du}+\left(\frac{u}{4}-\frac{s^2}{4u}\right)\xi_{ns} = i\left(n+\frac{s+1}{2}\right)\xi_{ns}, \tag{3.03}$$

The function $\xi_{ns}(u)$ is the regular solution, while $\eta_{ns}(u)$ has a singularity at $u=0$. The functions $\xi_{ns}(u)$ and $\eta_{ns}(u)$ satisfy identical recurrence relations that follow from (2.19) – (2.22). We have

$$2u\frac{\partial\xi_{ns}}{\partial u}+\xi_{ns} = (n+s+1)\xi_{n+1,s}-n\xi_{n-1,s}, \tag{3.04}$$

$$(2n+s+1+iu)\xi_{ns} = (n+s+1)\xi_{n+1,s}+n\xi_{n-1,s}. \tag{3.05}$$

On the other hand

$$\xi_{ns}-\xi_{n-1,s} = i\sqrt{u}\cdot\xi_{n-1,s+1}, \tag{3.06}$$

$$(n+s)\xi_{ns}-n\xi_{n-1,s} = \sqrt{u}\cdot\xi_{n,s-1}. \tag{3.07}$$

Combining the preceding relations, we obtain

$$\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)u^{-\frac{s}{2}}\xi_{ns}(u) = i(n+s+1)u^{-\frac{s+1}{2}}\xi_{n,s+1}(u), \tag{3.08}$$

$$\left(\frac{\partial}{\partial u}-\frac{i}{2}\right)u^{-\frac{s}{2}}\xi_{ns}(u) = inu^{-\frac{s+1}{2}}\xi_{n-1,s+1}(u). \tag{3.09}$$

The general expressions for the functions $\xi_{ns}(u)$ and $\eta_{ns}(u)$ are

$$\xi_{ns}(u) = e^{i\frac{u}{2}}\frac{u^{\frac{s}{2}}}{\Gamma(s+1)}F(-n,\ s+1,\ -iu), \tag{3.10}$$

$$\eta_{ns}(u) = -e^{i\frac{s\pi}{2}} \frac{n!}{\Gamma(n+s+1)} e^{-i\frac{u}{2}} u^{-\frac{s}{2}} \times$$

$$\times \int_0^\infty e^{-x} x^{n+s}(x+iu)^{-n-1} \, dx, \tag{3.11}$$

where F is the series (3.19) which reduces in our case to a polynomial of nth degree.

When s is an integer, the functions ξ_{ns} and η_{ns} can be expressed, with help of (3.06), in terms of ξ_{n0} and η_{n0}. The latter functions are expressible in terms of Laguerre's polynomials

$$L_n(x) = e^x \frac{d^n}{dx^n} (e^{-x} x^n) \tag{3.12}$$

and the sine and cosine integrals.

Putting for brevity

$$\xi_{n0}(u) = \xi_n(u); \quad \eta_{n0}(u) = \eta_n(u), \tag{3.13}$$

we have

$$\xi_n(u) = \frac{1}{n!} e^{i\frac{u}{2}} L_n(-iu), \tag{3.14}$$

$$\eta_n(u) = -\frac{1}{n!} e^{-i\frac{u}{2}} \int_0^\infty e^{-x} \frac{L_n(x) \, dx}{x+iu}. \tag{3.15}$$

Formula (3.15) is obtained from (3.11) by performing n partial integrations and by using the relation (3.12). The formula for $\eta_n(u)$ can be written in the form

$$\eta_n(u) = -\frac{1}{n!} e^{-i\frac{u}{2}} L_n(-iu) \int_0^\infty \frac{e^{-x} \, dx}{x+iu} -$$

$$-\frac{1}{n!} e^{-i\frac{u}{2}} \int_0^\infty e^{-x} \frac{L_n(x) - L_n(-iu)}{x+iu} \, dx. \tag{3.16}$$

The integral in the first term of (3.16) can be expressed using the sine and cosine integrals

$$\int_0^\infty \frac{e^{-x} \, dx}{x+iu} = e^{iu} \int_u^\infty e^{-iu'} \frac{du'}{u'} = e^{iu} \left(-\operatorname{Ci}(u) + i \operatorname{Si}(u) - i\frac{\pi}{2} \right), \tag{3.17}$$

and the integral in the second term is a polynomial in u.

The first few functions $\xi_n(u)$, $\eta_n(u)$ have the following explicit forms

$$\left.\begin{aligned}
\xi_0(u) &= e^{i\frac{u}{2}},\\
\xi_1(u) &= e^{i\frac{u}{2}}(1+iu),\\
\xi_2(u) &= e^{i\frac{u}{2}}\left(1+2iu-\frac{1}{2}u^2\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3.18}$$

$$\left.\begin{aligned}
\eta_0(u) &= e^{i\frac{u}{2}}\left[\mathrm{Ci}\,(u)-i\,\mathrm{Si}\,(u)+i\,\frac{\pi}{2}\right],\\
\eta_1(u) &= (1+iu)\,\eta_0(u)+e^{-i\frac{u}{2}},\\
\eta_2(u) &= \left(1+2iu-\frac{1}{2}u^2\right)\eta_0(u)+e^{-i\frac{u}{2}}\left(\frac{3}{2}+\frac{iu}{2}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3.19}$$

The subsequent functions can be expressed through the former with help of the recurrence relations (3.05) which take in our case the form

$$\left.\begin{aligned}
(2n+1+iu)\xi_n(u) &= n\xi_{n-1}(u)+(n+1)\xi_{n+1}(u),\\
(2n+1+iu)\eta_n(u) &= n\eta_{n-1}(u)+(n+1)\eta_{n+1}(u).
\end{aligned}\right\} \tag{3.20}$$

Asymptotic expressions for the functions $\xi_n(u)$, $\eta_n(u)$ valid for large values of n follow from the general formulae (2.25) – (2.28). We have

$$\xi_n(u) = J_0\left((1-i)\sqrt{\{(2n+1)u\}}\right), \tag{3.21}$$

$$\eta_n(u) = i\pi H_0^{(2)}\left((1-i)\sqrt{\{(2n+1)u\}}\right). \tag{3.22}$$

It is seen that, with increasing n, the functions $\xi_n(u)$ increase and the functions $\eta_n(u)$ decrease in absolute value.

4. *Expansion of a Point Singularity in Parabolic Functions*

When solving the problem of a radiating dipole located at the focus of a paraboloid of revolution, it is necessary to expand in parabolic functions the expression

$$\Pi = \frac{e^{ikR}}{kR} = \frac{2}{u+v}\,e^{i\frac{u+v}{2}}. \tag{4.01}$$

Since this expression is independent on φ, it is clear that only functions with $s=0$ will occur in the expansion. Having in mind the exponential in (4.01), it is natural to write the expansion in the form

$$\frac{2}{u+v}e^{i\frac{u+v}{2}} = \int_{-\infty}^{+\infty} f(t)\zeta_1(u, t)\zeta_1(v, -t)\, dt, \tag{4.02}$$

where $f(t)$ is a function to be determined.

Consider the following integral representations of the functions $\zeta_1(u, t)$ and $\zeta_2(v, -t)$ (they are a consequence of (2.11) and (2.12)):

$$e^{\frac{\pi}{2}(t+i)}\Gamma\left(\frac{1-it}{2}\right)e^{-\frac{iu}{2}}\zeta_1(u, t)$$
$$= \int_0^{\infty} e^{iup}p^{-\frac{1}{2}-\frac{it}{2}}(1+p)^{-\frac{1}{2}+\frac{it}{2}}\, dp; \tag{4.03}$$

$$e^{\frac{\pi}{2}(-t+i)}\Gamma\left(\frac{1+it}{2}\right)e^{-i\frac{v}{2}}\zeta_1(v, -t)$$
$$= \int_0^{\infty} e^{ivq}q^{-\frac{1}{2}+\frac{it}{2}}(1+q)^{-\frac{1}{2}-\frac{it}{2}}\, dq. \tag{4.04}$$

We multiply these expressions together and integrate the product with respect to t from $-\infty$ to $+\infty$. We thus obtain the integral

$$I = -\int_{-\infty}^{+\infty}\Gamma\left(\frac{1-it}{2}\right)\Gamma\left(\frac{1+it}{2}\right)e^{-i\frac{u+v}{2}}\zeta_1(u, t)\zeta_1(v, -t)\, dt$$
$$= \int_0^{\infty}\int_0^{\infty} e^{i(up+vq)}f(p, q)\, dp\, dq, \tag{4.05}$$

where

$$f(p, q) = \frac{1}{\sqrt{\{pq(1+p)(1+q)\}}}\cdot\int_{-\infty}^{+\infty} dt\left(\frac{1+p}{1+q}\cdot\frac{q}{p}\right)^{i\frac{t}{2}}. \tag{4.06}$$

Introducing the delta-function of Dirac

$$\delta(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ixt}\, dt, \tag{4.07}$$

having the well-known property that

$$\delta[\varphi(p)-\varphi(q)] = \frac{1}{\sqrt{\{|\varphi'(p)\varphi'(q)|\}}}\delta(p-q), \tag{4.08}$$

for monotonic $\varphi(p)$, we can interpret the expression (3.06) as

$$f(p, q) = 4\pi\delta(p-q). \tag{4.09}$$

Inserting this value of $f(p, q)$ into (4.05), we obtain an integral which, in turn, can be interpreted as

$$I = 4\pi\int_0^\infty e^{i(u+v)p}\,dp = \frac{4\pi i}{u+v}+4\pi^2\delta(u+v), \tag{4.10}$$

Since $U+v>0$ in our case, the term $\delta(u+v)$ can be dropped.

Equating (4.05) and (4.10) and making use of the equality

$$\Gamma\left(\frac{1+it}{2}\right)\Gamma\left(\frac{1-it}{2}\right) = \frac{\pi}{\cosh\frac{\pi t}{2}}, \tag{4.11}$$

we can write our result in the form

$$\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dt}{\cosh\frac{\pi t}{2}}\zeta_1(u, t)\zeta_1(v, -t) = \frac{2}{u+v}e^{i\frac{u+v}{2}}. \tag{4.12}$$

The argument, leading to (4.12), was, perhaps, not quite rigorous, but the result can be proved in a completely rigorous way. In order to do this, one can consider the identity

$$i(u+v)\zeta_1(u, t)\zeta_1(v, -t) = F(t)+F(t+2i), \tag{4.13}$$

with

$$F(t) = \zeta_1(u, t-2i)\zeta_1(v, -t)+\frac{1}{4}(1+it)^2\zeta_1(u, t)\zeta_1(v, -t+2i). \tag{4.14}$$

This identity follows immediately from (2.25). Using the equation

$$\cosh\frac{\pi t}{2} = -\cosh\frac{\pi(t+2i)}{2}, \tag{4.15}$$

we can write

$$\int_{-\infty}^{+\infty}\frac{i(u+v)}{\cosh\frac{\pi t}{2}}\zeta_1(u, t)\zeta_1(v, -t)\,dt$$

$$= \int_{-\infty}^{+\infty}\frac{F(t)}{\cosh\frac{\pi t}{2}}dt-\int_{-\infty}^{+\infty}\frac{F(t+2i)}{\cosh\frac{\pi}{2}(t+2i)}dt = 4F(i), \tag{4.16}$$

since the difference of the integrals is equal to the residue at the point $t=i$. But from (3.01) and (3.18) we have

$$F(i) = \zeta_1(u, -i)\zeta_1(v, -i) = \xi_0(u)\xi_0(v) = e^{i\frac{u+v}{2}}. \tag{4.17}$$

Inserting this into (4.16), we are led again to the equation

$$\frac{2}{u+v} e^{i\frac{u+v}{2}} = \frac{i}{2} \int_{-\infty}^{+\infty} \frac{dt}{\cosh \frac{\pi t}{2}} \zeta_1(u, t)\zeta_1(v, -t). \tag{4.12}$$

In spite of the fact that the parabolic functions involved in the expansion (4.12) have singularities on the axis of the paraboloid, the whole expression (4.12) remains finite on the axis.

From the integral representation of a point singularity a representation in series form can be easily obtained. For this purpose it is sufficient to calculate the integral as a sum of residues either at the points $t = -(2n+1)i$ or at the points $t = +(2n+1)i$. In the first case we obtain the series

$$\frac{2}{u+v} e^{i\frac{u+v}{2}} = 2i \sum_{n=0}^{\infty} \xi_n(u)\bar{\eta}_n(v), \tag{4.18}$$

and in the second case the series

$$\frac{2}{u+v} e^{i\frac{n+v}{2}} = 2i \sum_{n=0}^{\infty} \bar{\eta}_n(u)\xi_n(v). \tag{4.19}$$

The regions of convergence of these series can be easily established by means of the asymptotic expressions (3.21) and (3.22). The series (4.18) converges for $u < v$, and the series (4.19) converges for $u > v$.

5. *The Expansion of a Plane Wave*

Consider a scalar plane wave

$$e^{i\Omega} = e^{ik(x \sin\delta + z \cos\delta)} \tag{5.01}$$

Let us find its expansion in parabolic functions discussed in Section 2. Using the formulae (1.06) connecting the Cartesian coordinates x, y with the parabolic ones, we can write the expression for the phase Ω in the form

$$\Omega = \frac{1}{2}(u-v)\cos\delta + \sqrt{(uv)}\sin\delta\cos\varphi. \tag{5.02}$$

Expanding the wave (5.01) in a Fourier series of cosines of multiples of φ,

we obtain

$$e^{i\Omega} = e^{\frac{i}{2}(u-v)\cos\delta}\left\{J_0\left(\sqrt{(uv)}\sin\delta\right) + 2\sum_{s=1}^{\infty} i^s J_s\left(\sqrt{(uv)}\sin\delta\right)\cos s\varphi\right\}, \tag{5.03}$$

where J_s are Bessel functions.

Each term of this expression is to be expanded in functions of the form (2.04). Putting for brevity

$$C_s = e^{\frac{i}{2}(u-v)\cos\delta} J_s\left(\sqrt{(uv)}\sin\delta\right) \tag{5.04}$$

we must have an equality of the form

$$C_s = \int_{-\infty}^{+\infty} \psi(u, s, t)\psi(v, s, -t)f(t)\,dt \tag{5.05}$$

(the right-hand side cannot involve the functions ζ, since the expression (5.04) remains finite if one or both of the variables u, v, vanish). Now, expressions (5.04) and (5.05) must be equal for all values of the variable v, including $v \to 0$. Multiplying both sides of this equality with $\Gamma(s+1)v^{-\frac{s}{2}}$ and going to the limit $v \to 0$, we obtain

$$\lim_{v\to 0} \Gamma(s+1)v^{-\frac{s}{2}}C_s = e^{\frac{i}{2}u\cos\delta}u^{\frac{s}{2}}\left(\frac{\sin\delta}{2}\right)^s$$
$$= \int_{-\infty}^{+\infty} \Gamma\left(\frac{s+1-it}{2}\right)\psi(u, s, t)f(t)\,dt. \tag{5.06}$$

To find $f(t)$ from this equation we can make use of (2.23) and insert in (5.06) the integral representation (2.07) of $\xi(u, s, t)$. Dropping the factor $u^{\frac{s}{2}}e^{-i\frac{u}{2}}$ on both sides we obtain

$$\left(\frac{\sin\delta}{2}\right)^s e^{\frac{i}{2}u(1+\cos\delta)} = \int_{-\infty}^{+\infty} f(t)\,dt \int_0^1 e^{iuz} z^{\frac{s-1+it}{2}}(1-z)^{\frac{s-1-it}{2}}\,dz. \tag{5.07}$$

This equation will be satisfied if we choose $f(t)$ so as to satisfy the equation

$$\int_{-\infty}^{+\infty} f(t) z^{\frac{s-1+it}{2}}(1-z)^{\frac{s-1-it}{2}}\,dt = \left(\frac{\sin\delta}{2}\right)^s \delta_1\left(z - \frac{1+\cos\delta}{2}\right), \tag{5.08}$$

where the Dirac function is denoted by δ_1, to avoid confusion with the angle δ. Proceeding as in Section 4, one can easily see that equation

(5.08) will be satisfied if we put

$$f(t) = \frac{1}{2\pi \sin\delta}\left(\tan\frac{\delta}{2}\right)^{it}. \tag{5.09}$$

Inserting this value of $f(t)$ into equation (5.08), we obtain

$$e^{i\frac{u}{2}\cos\delta}\, u^{\frac{s}{2}}\left(\frac{\sin\delta}{2}\right)^{s} = \frac{1}{2\pi\sin\delta}\int_{-\infty}^{+\infty}\Gamma\left(\frac{s+1-it}{2}\right)\psi(u, s, t)\left(\tan\frac{\delta}{2}\right)^{it}\mathrm{d}t. \tag{5.10}$$

The same value of $f(t)$ inserted into (5.05) yields the more general equation

$$e^{\frac{i}{2}(u-v)\cos\delta}\, J_s(\sqrt{(uv)}\sin\delta)$$
$$= \frac{1}{2\pi\sin\delta}\int_{-\infty}^{+\infty}\psi(u, s, t)\psi(v, s, -t)\left(\tan\frac{\delta}{2}\right)^{it}\mathrm{d}t. \tag{5.11}$$

The proof of the relation (5.11) given above is not quite rigorous since we have used Dirac's delta-function. The result may be checked, however, by a straightforward calculation. One has only to insert into (5.11) the series expression for ψ resulting from (2.23) and (2.08), and to integrate term by term using the formula

$$\frac{1}{2\pi\sin\delta}\int_{-\infty}^{+\infty}\Gamma\left(\frac{s+1+it}{2}+k\right)\Gamma\left(\frac{s+1-it}{2}+l\right)\left(\tan\frac{\delta}{2}\right)^{it}\mathrm{d}t$$
$$= \Gamma(s+1+k+l)\left(\sin\frac{\delta}{2}\right)^{s+2l}\left(\cos\frac{\delta}{2}\right)^{s+2k}. \tag{5.12}$$

The double series thus obtained proves to be identical with the series for the left-hand side of (5.11).

Formula (5.11) holds not only for non-negative integral values of s, but also for all values of s such that the real part of $s+1$ is positive. If $Re(s+1)<0$, formula (5.11) still holds, provided an integration path is taken for which relation (5.12) is valid.

Our final result, namely the expansion of a plane wave in parabolic functions, can be written in the form

$$e^{i\Omega} = e^{\frac{i}{2}(u-v)\cos\delta+i\sqrt{(uv)}\sin\delta\cos\varphi}$$
$$= \frac{1}{2\pi\sin\delta}\int_{-\infty}^{+\infty}\Big\{\psi(u, 0, t)\psi(v, 0, -t)+$$
$$+2\sum_{s=1}^{\infty} i^s\psi(u, s, t)\psi(v, s, -t)\cos s\varphi\Big\}\left(\tan\frac{\delta}{2}\right)^{it}\mathrm{d}t. \tag{5.13}$$

6. *Maxwell's Equations and Parabolic Potentials*

We now turn from the scalar wave equation to Maxwell's equations. The time dependence of all field components will be assumed in the form of a factor $e^{-i\omega t}$, with $\omega = c\,k$, and in the following this factor will be not written out explicitly.

The Maxwell equations for the covariant field components in parabolic coordinates (see (1.09)) are of the form

$$\left.\begin{aligned} \frac{\partial E_\varphi}{\partial v} - \frac{\partial E_v}{\partial \varphi} &= iuH_u, \\ \frac{\partial E_u}{\partial \varphi} - \frac{\partial E_\varphi}{\partial u} &= ivH_v, \\ \frac{\partial E_v}{\partial u} - \frac{\partial E_u}{\partial v} &= i\frac{u+v}{4uv} H_\varphi; \end{aligned}\right\} \tag{6.01}$$

$$\left.\begin{aligned} \frac{\partial H_\varphi}{\partial v} - \frac{\partial H_v}{\partial \varphi} &= -iuE_u, \\ \frac{\partial H_u}{\partial \varphi} - \frac{\partial H_\varphi}{\partial u} &= -ivE_v, \\ \frac{\partial H_v}{\partial u} - \frac{\partial H_u}{\partial v} &= -i\frac{u+v}{4uv} E_\varphi, \end{aligned}\right\} \tag{6.02}$$

For any problem with boundary values prescribed on a given surface, the simplicity of the solution depends in a high degree on the appropriate choice of the potentials or of the auxiliary functions in terms of which the field is expressed. For Cartesian coordinates and a plane surface (the xy-plane) it is expedient to express the field in terms of two potentials, Φ and Ψ, connected with the Hertz vector. The expressions are of the form

$$\left.\begin{aligned} E_x &= \frac{\partial^2 \Psi}{\partial x\,\partial z} - ik\frac{\partial \Phi}{\partial y}; & H_x &= -\frac{\partial^2 \Phi}{\partial x\,\partial z} - ik\frac{\partial \Psi}{\partial y}, \\ E_y &= \frac{\partial^2 \Psi}{\partial y\,\partial z} + ik\frac{\partial \Phi}{\partial x}; & H_y &= -\frac{\partial^2 \Phi}{\partial y\,\partial z} + ik\frac{\partial \Psi}{\partial x}, \\ E_z &= \frac{\partial^2 \Psi}{\partial z^2} + k^2\Psi; & H_z &= \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2}. \end{aligned}\right\} \tag{6.03}$$

The most convenient potentials for spherical coordinates R, ϑ, φ are the Debye potentials U, V, in terms of which the field is expressed according

to the formulae

$$\left.\begin{aligned} E_R &= k^2RU+\frac{\partial^2(RU)}{\partial R^2}; & iH_R &= k^2RV+\frac{\partial^2(RV)}{\partial R}, \\ E_\vartheta &= \frac{\partial^2(RU)}{\partial R\,\partial\vartheta}+\frac{k}{\sin\vartheta}\frac{\partial(RV)}{\partial\varphi}; & iH_\vartheta &= \frac{\partial^2(RV)}{\partial R\,\partial\vartheta}+\frac{k}{\sin\vartheta}\frac{\partial(RU)}{\partial\varphi}, \\ E_\varphi &= \frac{\partial^2(RU)}{\partial R\,\partial\varphi}-k\sin\vartheta\frac{\partial(RV)}{\partial\vartheta}; & iH_\varphi &= \frac{\partial^2(RV)}{\partial R\,\partial\varphi}-k\sin\vartheta\frac{\partial(RU)}{\partial\vartheta}. \end{aligned}\right\} \tag{6.04}$$

The potentials Φ, Ψ, as well as the potentials U, V must satisfy the scalar wave equation (2.02).

In the case of parabolic coordinates it is most convenient to introduce potentials connected with the separate Fourier components of the field with respect to the angle φ, and not with the total field.

Let us first express the parabolic field components in terms of Debye's potentials. Applying to (6.04) the transformation formulae connecting the spherical components with the parabolic ones we obtain

$$\left.\begin{aligned} E_u &= \frac{1}{4}(u+v)U+\frac{\partial}{\partial u}(MU)-\frac{u+v}{4u}\frac{\partial V}{\partial\varphi}, \\ E_v &= \frac{1}{4}(u+v)U+\frac{\partial}{\partial v}(MU)+\frac{u+v}{4v}\frac{\partial V}{\partial\varphi}, \\ E_\varphi &= \frac{\partial}{\partial\varphi}(MU)+uv\left(\frac{\partial V}{\partial u}-\frac{\partial V}{\partial v}\right), \end{aligned}\right\} \tag{6.05}$$

where we have put for brevity

$$MU = u\frac{\partial U}{\partial u}+v\frac{\partial U}{\partial u}+U. \tag{6.06}$$

The quantities iH_u, iH_v, iH_φ are obtained from equations (6.05) by interchanging U with V.

The Debye potentials can be expanded in Fourier series of the form

$$\left.\begin{aligned} U &= \frac{1}{2}U^{\circ}+\sum_{s=1}^{\infty}U^{(s)}\cos s\varphi, \\ V &= \sum_{s=1}^{\infty}V^{(s)}\sin s\varphi. \end{aligned}\right\} \tag{6.07}$$

It is easily seen that from (6.07) cosine series for E_u, E_v, H_φ and sine series

for H_u, H_v, E_φ follow. We have

$$\left.\begin{aligned} E_u &= \frac{1}{2}E_u^{\circ}+\sum_{s=1}^{\infty}E_u^{(s)}\cos s\varphi; \quad E_v = \ldots; \quad H_\varphi = \ldots, \\ H_u &= \sum_{s=1}^{\infty}H_u^{(s)}\sin s\varphi; \quad H_v = \ldots; \quad E_\varphi = \ldots, \end{aligned}\right\} \tag{6.08}$$

where the rows of dots in the first line indicate cosine series and those in the second line — sine series. The coefficients of the series (6.08) are expressed in terms of $U^{(s)}$, $V^{(s)}$ according to the formulae

$$\left.\begin{aligned} E_u^{(s)} &= \frac{1}{4}(u+v)U^{(s)}+\frac{\partial}{\partial u}(MU^{(s)})-\frac{u+v}{4u}sV^{(s)} \\ E_v^{(s)} &= \frac{1}{4}(u+v)U^{(s)}+\frac{\partial}{\partial v}(MU^{(s)})+\frac{u+v}{4v}sV^{(s)} \\ E_\varphi^{(s)} &= -sMU^{(s)}+uv\left(\frac{\partial V^{(s)}}{\partial u}-\frac{\partial V^{(s)}}{\partial v}\right) \end{aligned}\right\} \tag{6.09}$$

$$\left.\begin{aligned} iH_u^{(s)} &= \frac{1}{4}(u+v)V^{(s)}+\frac{\partial}{\partial u}(MV^{(s)})+\frac{u+v}{4u}sU^{(s)} \\ iH_v^{(s)} &= \frac{1}{4}(u+v)V^{(s)}+\frac{\partial}{\partial v}(MV^{(s)})-\frac{u+v}{4v}sU^{(s)} \\ iH_\varphi^{(s)} &= sMV^{(s)}+uv\left(\frac{\partial U^{(s)}}{\partial u}-\frac{\partial U^{(s)}}{\partial v}\right). \end{aligned}\right\} \tag{6.10}$$

Since the functions U, V satisfy the scalar wave equation, their Fourier components must satisfy equations of the form

$$\left(L_u+L_v-\frac{s^2}{4u}-\frac{s^2}{4v}\right)U^{(s)} = 0. \tag{6.11}$$

where the symbols L_u, L_v denote the operators

$$\left.\begin{aligned} L_u &= u\frac{\partial^2}{\partial u^2}+\frac{\partial}{\partial u}+\frac{1}{4}u \\ L_v &= v\frac{\partial^2}{\partial v^2}+\frac{\partial}{\partial v}+\frac{1}{4}v. \end{aligned}\right\} \tag{6.12}$$

We introduce the quantities P_s°, Q_s° according to the formulae

$$U^{(s)} = \sqrt{(uv)^s}P_s^{\circ}; \qquad V^{(s)} = \sqrt{(uv)^s}Q_s^{\circ} \tag{6.13}$$

These quantities satisfy equations of the form

$$\left(L_u+L_v+s\frac{\partial}{\partial u}+s\frac{\partial}{\partial v}\right)P_s^\circ = 0. \tag{6.14}$$

Finally, we put

$$P_s^\circ = \frac{1}{u+v}\left(\frac{\partial P_{s-1}}{\partial u}+\frac{\partial P_{s-1}}{\partial v}\right); \qquad Q_s^\circ = \frac{1}{u+v}\left(\frac{\partial Q_{s-1}}{\partial u}+\frac{\partial Q_{s-1}}{\partial v}\right). \tag{6.15}$$

It is easily verified that if the functions P_{s-1}, Q_{s-1} satisfy equations of the form

$$\left[L_u+L_v+(s-1)\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\right]P_{s-1} = 0, \tag{6.16}$$

then P_s°, Q_s° will satisfy equations of the form (6.14). In other words, P_s, Q_s satisfy the same equations as P_s°, Q_s°.

Debye's potentials and the field can be expressed directly through P_{s-1} and Q_{s-1}. We have

$$U^{(s)} = \frac{\sqrt{(uv)^s}}{u+v}\left(\frac{\partial P_{s-1}}{\partial u}+\frac{\partial P_{s-1}}{\partial v}\right) \tag{6.17}$$

$$V^{(s)} = \frac{\sqrt{(uv)^s}}{u+v}\left(\frac{\partial Q_{s-1}}{\partial u}+\frac{\partial Q_{s-1}}{\partial v}\right). \tag{6.18}$$

Calculating the quantity $MU^{(s)}$ and using (6.16), we obtain

$$MU^{(s)} = \sqrt{(uv)^s}\left(\frac{\partial^2 P_{s-1}}{\partial u\,\partial v}-\frac{1}{4}P_{s-1}\right). \tag{6.19}$$

Inserting these expressions into (6.09), we obtain for the electric field the expressions

$$\left.\begin{aligned}
2uE_u^{(s)} &= \sqrt{(uv)^s}\left\{2\frac{\partial}{\partial v}P_{s-1}^*-s\frac{\partial^2 P_{s-1}}{\partial u\,\partial v}-\right.\\
&\qquad\left.-\frac{s}{4}P_{s-1}-\frac{s}{2}\frac{\partial Q_{s-1}}{\partial u}-\frac{s}{2}\frac{\partial Q_{s-1}}{\partial v}\right\}\\
2vE_v^{(s)} &= \sqrt{(uv)^{(s)}}\left\{-2\frac{\partial}{\partial u}P_{s-1}^*-s\frac{\partial^2 P_{s-1}}{\partial u\,\partial v}-\right.\\
&\qquad\left.-\frac{s}{4}P_{s-1}+\frac{s}{2}\frac{\partial Q_{s-1}}{\partial u}+\frac{s}{2}\frac{\partial Q_{s-1}}{\partial v}\right\}\\
E_\varphi^{(s)} &= \sqrt{(uv)^s}\left\{Q_{s-1}^*-\frac{s}{2}\frac{\partial Q_{s-1}}{\partial u}+\right.\\
&\qquad\left.+\frac{s}{2}\frac{\partial Q_{s-1}}{\partial v}-s\frac{\partial^2 P_{s-1}}{\partial u\,\partial v}+\frac{s}{4}P_{s-1}\right\}.
\end{aligned}\right\} \tag{6.20}$$

Similar expressions are obtained for the magnetic field, namely

$$\left.\begin{aligned}
2iuH_u^{(s)} &= \sqrt{(uv)^s}\left\{2\frac{\partial}{\partial v}Q_{s-1}^{*}-s\frac{\partial^2 Q_{s-1}}{\partial u\,\partial v}-\right.\\
&\quad\left.-\frac{s}{4}Q_{s-1}+\frac{s}{2}\frac{\partial P_{s-1}}{\partial u}+\frac{s}{2}\frac{\partial P_{s-1}}{\partial v}\right\}\\
2ivH_v^{(s)} &= \sqrt{(uv)^s}\left\{-2\frac{\partial}{\partial u}Q_{s-1}^{*}-s\frac{\partial^2 Q_{s-1}}{\partial u\,\partial v}-\right.\\
&\quad\left.-\frac{s}{4}Q_{s-1}-\frac{s}{2}\frac{\partial P_{s-1}}{\partial u}-\frac{s}{2}\frac{\partial P_{s-1}}{\partial v}\right\}\\
iH_\varphi^{(s)} &= \sqrt{(uv)^s}\left\{P_{s-1}^{*}-\frac{s}{2}\frac{\partial P_{s-1}}{\partial u}+\right.\\
&\quad\left.+\frac{s}{2}\frac{\partial P_{s-1}}{\partial v}+s\frac{\partial^2 Q_{s-1}}{\partial u\,\partial v}-\frac{s}{4}Q_{s-1}\right\}.
\end{aligned}\right\} \tag{6.21}$$

In these formulae we have introduced the abbreviations P_{s-1}^{*}, Q_{s-1}^{*} for the function

$$\begin{aligned}
P_{s-1}^{*} &= u\frac{\partial^2 P_{s-1}}{\partial u^2}+s\frac{\partial P_{s-1}}{\partial u}+\frac{1}{4}uP_{s-1}\\
&= -v\frac{\partial^2 P_{s-1}}{\partial v^2}-s\frac{\partial P_{s-1}}{\partial v}-\frac{1}{4}vP_{s-1},
\end{aligned} \tag{6.22}$$

and for the function Q_{s-1}^{*} connected with Q_{s-1} by a similar relation (with P and Q interchanged). In the case when P_{s-1} and Q_{s-1} are products of a function of u and a function of v, the quantity P_{s-1}^{*} will be simply proportional to P_{s-1} and Q_{s-1}^{*} proportional to Q_{s-1}.

The transformed expressions (6.20) and (6.21) have the advantage over the original expressions (6.09) and (6.10) that they allow a simple formulation of boundary conditions on the surface of the paraboloid v=constant. Indeed, if we put

$$2\frac{\partial P_{s-1}}{\partial v}+Q_{s-1} = A \tag{6.23}$$

$$4Q_{s-1}^{*}+2s\frac{\partial Q_{s-1}}{\partial v}+sP_{s-1} = B, \tag{6.24}$$

then the tangential components of the electric field and the normal compo-

nent of the magnetic field will be equal to

$$\left.\begin{aligned} 2uE_u^{(s)} &= \sqrt{(uv)^s}\left(u\frac{\partial^2 A}{\partial u^2}+\frac{1}{4}uA+\frac{s}{2}\frac{\partial A}{\partial u}-\frac{1}{4}B\right) \\ E_\varphi^{(s)} &= \sqrt{(uv)^s}\left(\frac{1}{4}B-\frac{s}{2}\frac{\partial A}{\partial u}\right) \\ 2ivH_v^{(s)} &= \sqrt{(uv)^s}\left(-\frac{s}{4}A-\frac{1}{2}\frac{\partial B}{\partial u}\right). \end{aligned}\right\} \tag{6.25}$$

These expressions must vanish on the surface of a perfectly conducting paraboloid. To ensure this, it is sufficient to impose the condition that for $v=v_0$ and for all values of u it should be $A=0$, $B=0$, or in more detail

$$2\frac{\partial P_{s-1}}{\partial v}+Q_{s-1} = 0 \tag{6.26}$$

$$4Q_{s-1}^*+2s\frac{\partial Q_{s-1}}{\partial v}+sP_{s-1} = 0. \tag{6.27}$$

It is essential that the left-hand sides of the boundary conditions (6.26) and (6.27) are obtained from the functions P_{s-1}, Q_{s-1} by applying operators not involving u (neither a multiplication by u, nor a differentiation with respect to u).† Therefore, if P_{s-1}, Q_{s-1} are expanded in a series (or an integral) of functions of u forming an orthogonal system, then the left-hand sides of the boundary conditions will represent expansions of the same type, and in order that they vanish it is sufficient to equate the coefficients of these expansions to zero.

If, instead of (6.20) and (6.21), we use expressions (6.09) and (6.10) we have to find the coefficients of the expansions of the functions $U^{(s)}$, $V^{(s)}$. The equations obtained for these coefficients would be not algebraic, but they would form a system of finite difference equations, which circumstance would considerably complicate our problem. Still greater complications arise, if the potentials Φ, Ψ of formulae (6.03) are used.

Thus, the introduction of the potentials P_s, Q_s permit us to avoid equations in finite differences and, therefore, to avoid the complicated formalism necessary for their solution.

† Let us remember that, according to (6.22), the quantity Q_{s-1}^* can be expressed in terms of Q_{s-1}, by the equation

$$Q_{s-1}^* = -v\frac{\partial^2 Q_{s-1}}{\partial v^2}-s\frac{\partial Q_{s-1}}{\partial v}-\frac{1}{4}vQ_{s-1}$$

7. *Transformation of the Expressions for the Field*

A slight inconvenience of the potentials P_s, Q_s is that the expressions (6.20), (6.21) for the field in terms of these potentials are somewhat complicated. This inconvenience can be avoided if four auxiliary functions C_{s-1}, D_{s-1}, F_s, G_s, are introduced, in terms of which the field is expressed in a more simple way. These functions have simple properties and are connected with the potentials P_{s-1}, Q_{s-1} by simple relations.

We put

$$C_{s-1} = P^*_{s-1} - \frac{s}{2} Q_{s-1} \tag{7.01}$$

$$D_{s-1} = Q^*_{s-1} + \frac{s}{2} P_{s-1} \tag{7.02}$$

$$F_s = \frac{\partial^2 P_{s-1}}{\partial u\, \partial v} + \frac{1}{4} P_{s-1} + \frac{1}{2} \frac{\partial Q_{s-1}}{\partial u} - \frac{1}{2} \frac{\partial Q_{s-1}}{\partial v} \tag{7.03}$$

$$G_s = \frac{\partial^2 Q_{s-1}}{\partial u\, \partial v} + \frac{1}{4} Q_{s-1} - \frac{1}{2} \frac{\partial P_{s-1}}{\partial u} + \frac{1}{2} \frac{\partial P_{s-1}}{\partial v}\,. \tag{7.04}$$

Expressions (6.20) and (6.21) for the field can be written thus:

$$\left.\begin{aligned} 2uE_u^{(s)} &= \sqrt{(uv)^s}\left(2\frac{\partial C_{s-1}}{\partial v} - sF_s\right) \\ 2vE_v^{(s)} &= \sqrt{(uv)^s}\left(-2\frac{\partial C_{s-1}}{\partial u} - sF_s\right) \\ E_\varphi^{(s)} &= \sqrt{(uv)^s}\,(D_{s-1} - sF_s) \end{aligned}\right\} \tag{7.05}$$

$$\left.\begin{aligned} 2iuH_u^{(s)} &= \sqrt{(uv)^s}\left(2\frac{\partial D_{s-1}}{\partial v} - sG_s\right) \\ 2ivH_v^{(s)} &= \sqrt{(uv)^s}\left(-2\frac{\partial D_{s-1}}{\partial u} - sG_s\right) \\ iH_\varphi^{(s)} &= \sqrt{(uv)^s}\,(C_{s-1} + sG_s). \end{aligned}\right\} \tag{7.06}$$

The functions C_s, D_s, F_s, G_s satisfy the same differential equation as P_s, Q_s, namely

$$\left(u\frac{\partial^2}{\partial u^2} + (s+1)\frac{\partial}{\partial u} + \frac{1}{4}u\right)F_s = -\left(v\frac{\partial^2}{\partial v^2} + (s+1)\frac{\partial}{\partial v} + \frac{1}{4}v\right)F_s\,. \tag{7.07}$$

Obviously, the functions C_{s-1}, D_{s-1} satisfy equation (7.07) with s replaced

by $s-1$. The relations between the functions $C_{s-1}, D_{s-1}, P_{s-1}, Q_{s-1}$ can be written in the form of equations

$$C_{s-1}+iD_{s-1} = \left(u\frac{\partial}{\partial u}+s-\frac{iu}{2}\right)\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)(P_{s-1}+iQ_{s-1})$$
$$= -\left(v\frac{\partial}{\partial v}+s+\frac{iv}{2}\right)\left(\frac{\partial}{\partial v}-\frac{i}{2}\right)(P_{s-1}+iQ_{s-1}) \tag{7.08}$$

and of other equations, obtained from (7.08) by interchanging i and $-i$ (as if the quantities P, Q, C, D, were real). On the other hand, from (7.03) and (7.04) there follows

$$F_s+iG_s = \left(\frac{\partial}{\partial v}-\frac{i}{2}\right)\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)(P_{s-1}+iQ_{s-1}) \tag{7.09}$$

and a similar equation with i and $-i$ interchanged. Comparison of (7.08) with (7.09) leads to two relations

$$\left.\begin{aligned}&\left(u\frac{\partial}{\partial u}+s-\frac{iu}{2}\right)(F_s+iG_s)\\ &\qquad\qquad = \left(\frac{\partial}{\partial v}-\frac{i}{2}\right)(C_{s-1}+iD_{s-1}),\\ &\left(v\frac{\partial}{\partial v}+s+\frac{iv}{2}\right)(F_s+iG_s)\\ &\qquad\qquad = -\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)(C_{s-1}+iD_{s-1})\end{aligned}\right\} \tag{7.10}$$

and to two relations with i and $-i$ interchanged. Separating (formally) the real and the imaginary parts, we obtain

$$\left(u\frac{\partial}{\partial u}+s\right)F_s+\frac{u}{2}G_s = \frac{\partial}{\partial v}C_{s-1}+\frac{1}{2}D_{s-1}, \tag{7.11}$$

$$\left(u\frac{\partial}{\partial u}+s\right)G_s-\frac{u}{2}F_s = \frac{\partial}{\partial v}D_{s-1}-\frac{1}{2}C_{s-1}, \tag{7.12}$$

$$\left(v\frac{\partial}{\partial v}+s\right)F_s-\frac{v}{2}G_s = -\frac{\partial}{\partial u}C_{s-1}+\frac{1}{2}D_{s-1}, \tag{7.13}$$

$$\left(v\frac{\partial}{\partial v}+s\right)G_s+\frac{v}{2}F_s = -\frac{\partial}{\partial u}D_{s-1}-\frac{1}{2}C_{s-1}. \tag{7.14}$$

The auxiliary functions thus introduced permit us to express the field in terms of the usual scalar and vector potentials, and also in terms of the

magnetic potentials. Indeed, the field with parabolic components

$$E_u = E_u^{(s)} \cos s\varphi; \quad E_v = E_v^{(s)} \cos s\varphi; \quad E_\varphi = E_\varphi^{(s)} \sin s\varphi, \tag{7.15}$$

$$H_u = H_u^{(s)} \sin s\varphi; \quad H_v = H_v^{(s)} \sin s\varphi; \quad H_\varphi = H_\varphi^{(s)} \cos s\varphi \tag{7.16}$$

can be expressed either in the form

$$\mathbf{E} = ik\mathbf{A} - \operatorname{grad} A_0; \quad \mathbf{H} = \operatorname{curl} \mathbf{A}, \tag{7.17}$$

with

$$\operatorname{div} \mathbf{A} = ikA_0, \tag{7.18}$$

or else in the form

$$\mathbf{E} = \operatorname{curl} \mathbf{B}; \quad \mathbf{H} = -ik\mathbf{B} + \operatorname{grad} B_0, \tag{7.19}$$

with

$$\operatorname{div} \mathbf{B} = ikB_0, \tag{7.20}$$

The electric potentials are equal to

$$\left.\begin{aligned} A_x &= iD_{s-1}\sqrt{(uv)^{s-1}} \cos (s-1)\varphi, \\ A_y &= -iD_{s-1}\sqrt{(uv)^{s-1}} \sin (s-1)\varphi, \\ A_z &= -iG_s\sqrt{(uv)^s} \cos s\varphi, \\ A_0 &= -F_s\sqrt{(uv)^s} \cos s\varphi, \end{aligned}\right\} \tag{7.21}$$

and the magnetic potentials are equal to

$$\left.\begin{aligned} B_x &= C_{s-1}\sqrt{(uv)^{s-1}} \sin (s-1)\varphi, \\ B_y &= C_{s-1}\sqrt{(uv)^{s-1}} \cos (s-1)\varphi, \\ B_z &= -F_s\sqrt{(uv)^s} \sin s\varphi, \\ B_0 &= -iG_s\sqrt{(uv)^s} \sin s\varphi. \end{aligned}\right\} \tag{7.22}$$

8. *Series for the Potentials and for the Auxiliary Functions*

When expressions for the potentials P_{s-1}, Q_{s-1} in the form of series of parabolic functions are known, one can easily deduce series of the same form for the quantities C_{s-1}, D_{s-1}, F_s, G_s. We introduce the functions

$$\chi_{ns}(u, v) = \frac{\Gamma(n+s+1)}{\Gamma(n+1)} (uv)^{-\frac{s}{2}} \xi_{ns}(u)\xi_{ns}(v). \tag{8.01}$$

As a consequence of the differential equation (3.03) they satisfy the relations

$$\left(u\frac{\partial}{\partial u}+s+1-\frac{iu}{2}\right)\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)\chi_{ns} = i(n+s+1)\chi_{ns}, \tag{8.02}$$

$$\left(u\frac{\partial}{\partial u}+s+1+\frac{iu}{2}\right)\left(\frac{\partial}{\partial u}-\frac{i}{2}\right)\chi_{ns} = in\chi_{ns} \tag{8.03}$$

and also, as a consequence of (3.08) and (3.09)

$$\left(\frac{\partial}{\partial v}-\frac{i}{2}\right)\left(\frac{\partial}{\partial u}+\frac{i}{2}\right)\chi_{ns} = (n+s+1)\chi_{n,s+1}, \tag{8.04}$$

$$\left(\frac{\partial}{\partial v}+\frac{i}{2}\right)\left(\frac{\partial}{\partial u}-\frac{i}{2}\right)\chi_{ns} = n\chi_{n-1,s+1}. \tag{8.05}$$

The same relations are satisfied by the functions χ_{ns}, if one or both functions ξ_{ns} are replaced by η_{ns}. In the following we mean by χ_{ns} one of the four functions thus obtained.

Let $s \geqslant 1$ and let the series for P_{s-1} and Q_{s-1} have the form

$$P_{s-1} = \sum_n p_n \chi_{n,s-1}, \tag{8.06}$$

$$Q_{s-1} = \sum_n q_n \chi_{n,s-1}. \tag{8.07}$$

Applying the formulae deduced above, we obtain

$$C_{s-1}+iD_{s-1} = i\sum_n (n+s)(p_n+iq_n)\chi_{n,s-1}, \tag{8.08}$$

$$C_{s-1}-iD_{s-1} = i\sum_n n(p_n-iq_n)\chi_{n,s-1}, \tag{8.09}$$

and also

$$F_s+iG_s = \sum_n (n+s)(p_n+iq_n)\chi_{n,s}, \tag{8.10}$$

$$F_s-iG_s = \sum_n n(p_n-iq_n)\chi_{n-1,s}. \tag{8.11}$$

The latter formula has a definite sense only if $p_0=iq_0$ because the quantity $n\chi_{n-1,s}$ is defined only for $n\neq 0$.

Similar formulae for the auxiliary functions are obtained if the series for the potentials P_s, Q_s proceed in functions that are conjugate imaginaries to χ_{ns}. Let the series for P_{s-1} and Q_{s-1} be of the form

$$P_{s-1} = \sum_n p_n \bar{\chi}_{n,s-1}, \tag{8.12}$$

$$Q_{s-1} = \sum_n q_n \bar{\chi}_{n,s-1}. \tag{8.13}$$

Then we will have

$$C_{s-1}+iD_{s-1} = -i\sum_n n(p_n+iq_n)\bar{\chi}_{n,s-1}, \tag{8.14}$$

$$C_{s-1}-iD_{s-1} = -i\sum_n (n+s)(p_n-iq_n)\bar{\chi}_{n,s-1}, \tag{8.15}$$

and also

$$F_s+iG_s = \sum_n n(p_n+iq_n)\bar{\chi}_{n-1,\,s}, \tag{8.16}$$

$$F_s-iG_s = \sum_n (n+s)\,(p_n-iq_n)\bar{\chi}_{ns}. \tag{8.17}$$

In conclusion we will derive formulae for the derivatives of the series proceeding in functions χ_{ns}. If r and z are cylindrical coordinates, we have

$$\frac{1}{k}\frac{\partial F}{\partial z} = \frac{2}{u+v}\left(u\frac{\partial F}{\partial u}-v\frac{\partial F}{\partial v}\right), \tag{8.18}$$

$$\frac{1}{k^2r}\frac{\partial F}{\partial r} = \frac{2}{u+v}\left(\frac{\partial F}{\partial u}+\frac{\partial F}{\partial v}\right). \tag{8.19}$$

If F is a series of the form

$$F = \sum_{n=0}^{\infty} a_n\chi_{ns}(u, v), \tag{8.20}$$

then the right-hand side of (8.18) will be, in general, a series of the same form

$$\frac{2}{u+v}\left(u\frac{\partial F}{\partial u}-v\frac{\partial F}{\partial v}\right) = i\sum_{n=0}^{\infty} b_n\chi_{ns}(u, v), \tag{8.21}$$

while the right-hand side of (8.19) will be a series proceeding in functions $\chi_{n,\,s+1}$ and can be written as

$$\frac{2}{u+v}\left(\frac{\partial F}{\partial u}+\frac{\partial F}{\partial v}\right) = 2\sum_{n=0}^{\infty} c_n\chi_{n,\,s+1}(u, v). \tag{8.22}$$

The reservation "in general" is necessary because our assertion applies without restriction only to functions χ_{ns} of the form (8.01), built up of functions ξ_{ns} only. If, however, χ_{ns} involves one or two functions η_{ns}, then formulae (8.21) and (8.22) are valid only with the supplementary condition $a_0=0$.

To find the connection between the coefficients a_n and b_n, we consider the expression $2\left(u\frac{\partial F}{\partial u}-v\frac{\partial F}{\partial v}\right)$ where F is the series (8.20). On the one hand we can apply to this expression the recurrence formula (3.04); on the other hand we obtain for the same quantity another expression if we multiply the series (8.21) by $u+v$ and use the relation (3.05). Equating both expressions for the quantity considered we obtain the required connection in the form

$$a_n+a_{n+1} = b_n-b_{n+1}. \tag{8.23}$$

Comparing the sum of the partial derivatives of the series (8.20) with respect to u and to v with the series (8.22) multiplied by $u+v$, and using the recurrence formulae (3.08) and (3.09), we obtain in a similar way

$$a_{n+1} = c_n - c_{n+1}. \tag{8.24}$$

Hence we may conclude that

$$b_{n+1} = c_n + c_{n+1} + \text{constant} \tag{8.25}$$

If χ_{ns} involves only the functions ξ_{ns}, then the series $\sum_{n=0}^{\infty} a_n$ must converge, and we have

$$c_n = \sum_{k=n+1}^{\infty} a_k; \quad b_n = a_n + 2 \sum_{k=n+1}^{\infty} a_k. \tag{8.26}$$

The value of the constant in (8.25) is zero.

Similar transformations can be applied to integral expansions, but we shall not deal with them here.

A Radiating Dipole at the Focus of a Paraboloid of Revolution

9. *Primary Field of a Radiating Dipole*

Consider a radiating dipole situated at the origin of coordinates and having a moment in the direction of the x-axis (at right angles to the axis of symmetry). The magnetic field of the dipole can be expressed, according to the formulae

$$H_x^\circ = 0; \quad H_y^\circ = \frac{\partial A_x^\circ}{\partial z}; \quad H_z^\circ = -\frac{\partial A_x^\circ}{\partial y} \tag{9.01}$$

in terms of a vector-potential having only one non-vanishing component

$$A_x^\circ = A = \frac{iC}{2kR} e^{ikR} = \frac{iC}{u+v} e^{i\frac{u+v}{2}}, \tag{9.02}$$

where C is a constant.

The covariant parabolic components of the magnetic field can be found from the formulae

$$\left.\begin{aligned} 2uH_u^\circ &= -2\sqrt{(uv)}\frac{\partial A}{\partial v}\sin\varphi, \\ 2vH_v^\circ &= 2\sqrt{(uv)}\frac{\partial A}{\partial u}\sin\varphi, \\ H_\varphi^\circ &= 2\sqrt{(uv)}\frac{1}{u+v}\left(u\frac{\partial A}{\partial u} - v\frac{\partial A}{\partial v}\right)\cos\varphi, \end{aligned}\right\} \tag{9.03}$$

and the components of the electric field can be obtained by applying Maxwell's equations (6.02) to the magnetic field (9.03). In the following we will require exact as well as approximate expressions for the primary field; they will be written down in full. We have

$$\left.\begin{aligned}
2uE_u^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\cos\varphi\left\{\frac{u-v}{(u+v)^2}+6i\,\frac{u-v}{(u+v)^3}+\right.\\
&\quad\left.+\frac{4i}{(u+v)^2}-\frac{8}{(u+v)^3}-\frac{12(u-v)}{(u+v)^4}\right\},\\
2vE_v^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\cos\varphi\left\{-\frac{u-v}{(u+v)^2}-6i\,\frac{u-v}{(u+v)^3}+\right.\\
&\quad\left.+\frac{4i}{(u+v)^2}-\frac{8}{(u+v)^3}+\frac{12(u-v)}{(u+v)^4}\right\},\\
E_\varphi^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\sin\varphi\left\{\frac{1}{u+v}+\frac{2i}{(u+v)^2}-\frac{4}{(u+v)^3}\right\}.
\end{aligned}\right\}\tag{9.04}$$

$$\left.\begin{aligned}
2uH_u^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\sin\varphi\left\{\frac{1}{u+v}+\frac{2i}{(u+v)^2}\right\},\\
2vH_v^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\sin\varphi\left\{-\frac{1}{u+v}-\frac{2i}{(u+v)^2}\right\},\\
H_\varphi^\circ &= C\sqrt{(uv)}\,e^{i\frac{u+v}{2}}\cos\varphi\left\{-\frac{(u-v)}{(u+v)^2}-\frac{2i(u-v)}{(u+v)^3}\right\}.
\end{aligned}\right\}\tag{9.05}$$

We must express, in terms of the potentials P, Q, the primary field of the dipole, and also the total field, including that of the reflected wave. The dependence of the total field on the angle φ will be the same as for the primary field; therefore, we must put $s=1$ in (6.20) and (6.21). Dropping the indices of P and Q, we can write the field expressions resulting from these formulae in the form

$$\left.\begin{aligned}
2uE_u &= \sqrt{(uv)}\left\{2\frac{\partial P^*}{\partial v}-\frac{\partial^2 P}{\partial u\,\partial v}-\frac{1}{4}P-\frac{1}{2}\frac{\partial Q}{\partial u}-\frac{1}{2}\frac{\partial Q}{\partial v}\right\}\cos\varphi,\\
2vE_v &= \sqrt{(uv)}\left\{-2\frac{\partial P^*}{\partial u}-\frac{\partial^2 P}{\partial u\,\partial v}-\frac{1}{4}P+\frac{1}{2}\frac{\partial Q}{\partial u}+\frac{1}{2}\frac{\partial Q}{\partial v}\right\}\cos\varphi,\\
E_\varphi &= \sqrt{(uv)}\left\{Q^*-\frac{1}{2}\frac{\partial Q}{\partial u}+\frac{1}{2}\frac{\partial Q}{\partial v}-\frac{\partial^2 P}{\partial u\,\partial v}+\frac{1}{4}P\right\}\sin\varphi,
\end{aligned}\right\}\tag{9.06}$$

$$\left.\begin{aligned}
2iuH_u &= \sqrt{(uv)}\left\{2\frac{\partial Q^*}{\partial u\partial v}-\frac{\partial^2 Q}{\partial u\partial v}-\frac{1}{4}Q+\frac{1}{2}\frac{\partial P}{\partial u}+\frac{1}{2}\frac{\partial P}{\partial v}\right\}\sin\varphi,\\
2ivH_v &= \sqrt{(uv)}\left\{-2\frac{\partial Q^*}{\partial u}-\frac{\partial^2 Q}{\partial u\,\partial v}-\frac{1}{4}Q-\frac{1}{2}\frac{\partial P}{\partial u}-\frac{1}{2}\frac{\partial P}{\partial v}\right\}\sin\varphi,\\
iH_\varphi &= \sqrt{(uv)}\left\{P^*-\frac{1}{2}\frac{\partial P}{\partial u}+\frac{1}{2}\frac{\partial P}{\partial v}+\frac{\partial^2 Q}{\partial u\,\partial v}-\frac{1}{4}Q\right\}\cos\varphi.
\end{aligned}\right\} \quad (9.07)$$

The functions P and Q in the above formulae are independent of φ and satisfy the scalar wave equation which can be written as

$$(L_u+L_v)P = 0; \quad L_u = u\frac{\partial^2}{\partial u^2}+\frac{\partial}{\partial u}+\frac{1}{4}u. \quad (9.08)$$

According to the general formulae (6.22), the functions P^*, Q^* are connected with P, Q by the relations

$$P^* = L_u P = -L_v P \quad (9.09)$$

$$Q^* = L_u Q = -L_v Q. \quad (9.10)$$

These functions also satisfy the scalar wave equation.

The expressions for the field are simplified if the auxiliary functions (7.01) – (7.04) are introduced. Performing the transformation we write for brevity

$$C_0 = S; \quad D_0 = T \quad (9.11)$$

and also keep the notation F_1 and G_1. We thus put

$$S = P^*-\frac{1}{2}Q, \quad (9.12)$$

$$T = Q^*+\frac{1}{2}P, \quad (9.13)$$

$$F_1 = \frac{\partial^2 P}{\partial u\,\partial v}+\frac{1}{4}P+\frac{1}{2}\frac{\partial Q}{\partial u}-\frac{1}{2}\frac{\partial Q}{\partial v}. \quad (9.14)$$

$$G_1 = \frac{\partial^2 Q}{\partial u\,\partial v}+\frac{1}{4}Q-\frac{1}{2}\frac{\partial P}{\partial u}+\frac{1}{2}\frac{\partial P}{\partial v}. \quad (9.15)$$

Using this notation, we have

$$\left.\begin{aligned}
2uE_u &= \sqrt{(uv)}\left(2\frac{\partial S}{\partial v}-F_1\right)\cos\varphi\\
2vE_v &= \sqrt{(uv)}\left(-2\frac{\partial S}{\partial u}-F_1\right)\cos\varphi\\
E_\varphi &= \sqrt{(uv)}\,(T-F_1)\sin\varphi
\end{aligned}\right\} \quad (9.16)$$

$$\left.\begin{aligned} 2iuH_u &= \sqrt{(uv)}\left(2\frac{\partial T}{\partial v}-G_1\right)\sin\varphi \\ 2ivH_v &= \sqrt{(uv)}\left(-2\frac{\partial T}{\partial u}-G_1\right)\sin\varphi \\ iH_\varphi &= \sqrt{(uv)}\,(S+G_1)\cos\varphi. \end{aligned}\right\} \tag{9.17}$$

Let us find the functions $P=P^\circ$, $Q=Q^\circ$ that correspond to the field of a free dipole (without the reflected wave). From (9.05) it follows that

$$RH_R^\circ = uH_u^\circ+vH_v^\circ = 0, \tag{9.18}$$

as it ought to be, since the magnetic field of a dipole has no radial component. For the primary field, the sum of the first two expressions (9.07) must therefore vanish. This will be the case if we put $Q^\circ=0$. As to the quantity P°, it is easy to see that the expressions (9.07) will coincide with (9.03) or with (9.05) if we put

$$P^\circ = \frac{2C}{u+v}\,e^{i\frac{u+v}{2}} = C\Pi, \tag{9.19}$$

where Π is the point singularity considered in Section 4. The auxiliary functions (9.12)–(9.15) which correspond to the values of P°, Q° just found will be equal to

$$S^\circ = C\,e^{i\frac{u+v}{2}}\left(\frac{2(u-v)}{(u+v)^3}-\frac{i(u-v)}{(u+v)^2}\right), \tag{9.20}$$

$$T^\circ = \frac{C}{u+v}\,e^{i\frac{u+v}{2}}, \tag{9.21}$$

$$F_1^\circ = Ce^{i\frac{u+v}{2}}\left(\frac{4}{(u+v)^3}-\frac{2i}{(u+v)^2}\right), \tag{9.22}$$

$$G_1^\circ = 0. \tag{9.23}$$

The usual electric potentials calculated from (7.21) with due account of the notations (9.11) will be

$$A_x^\circ = \frac{iC}{u+v}\,e^{i\frac{u+v}{2}}; \quad A_y^\circ = 0; \quad A_z^\circ = 0, \tag{9.24}$$

$$A_0^\circ = -C\,e^{i\frac{u+v}{2}}\left(\frac{4}{(u+v)^3}-\frac{2i}{(u+v)^2}\right)\sqrt{(uv)}\cos\varphi, \tag{9.25}$$

which is the same as (9.02).

10. *The Field of the Reflected Wave Expressed in the Form of Integrals*

We now proceed to find the expressions for the reflected wave potentials in the form of integrals. For the potential P° of the primary wave we have, according to (4.12), the integral representation

$$P^\circ = \frac{2C}{u+v} e^{i\frac{u+v}{2}} = \frac{iC}{2} \int_{-\infty}^{+\infty} \frac{dt}{\cosh\frac{\pi t}{2}} \zeta_1(u, t)\zeta_1(v, -t). \tag{10.01}$$

For the total field we write

$$\left.\begin{aligned} P &= P^\circ + P' \\ Q &= Q', \end{aligned}\right\} \tag{10.02}$$

where P' and Q' correspond to the reflected wave. From considerations based on geometrical optics it is clear that the reflected wave must have the phase factor

$$e^{ikz} = e^{i\frac{u-v}{2}}. \tag{10.03}$$

Now, this is the factor appearing in the asymptotic expression of the function

$$\zeta_1(u, t)\zeta_2(v, -t) = (uv)^{-\frac{1}{2}+\frac{it}{2}} \cdot e^{i\frac{u-v}{2}} (1 + \ldots). \tag{10.04}$$

The integrals representing the reflected wave must therefore be of the form

$$P' = \frac{iC}{2} \int_{-\infty}^{+\infty} \frac{p(t)}{\cosh\frac{\pi t}{2}} \zeta_1(u, t)\zeta_2(v, -t)\, dt, \tag{10.05}$$

$$Q' = \frac{iC}{2} \int_{-\infty}^{+\infty} \frac{q(t)}{\cosh\frac{\pi t}{2}} \zeta_1(u, t)\zeta_2(v, -t)\, dt, \tag{10.06}$$

where the unknown functions $p(t)$, $q(t)$ are to be determined from the boundary conditions. According to the general formulae (6.28), (6.27), valid for an absolutely reflecting paraboloid, these conditions are of the form

$$2\frac{\partial P'}{\partial v} + Q' = -2\frac{\partial P^\circ}{\partial v} \text{(for } v = v_0), \tag{10.07}$$

$$4Q'^* + 2\frac{\partial Q'}{\partial v} + P' = -P^\circ \text{ (for } v = v_0). \tag{10.08}$$

The quantity Q'^* in (10.08) is equal to

$$Q'^* = L_u Q' = -\frac{iC}{4}\int_{-\infty}^{+\infty} \frac{tq(t)}{\cosh\frac{\pi t}{2}}\,\zeta_1(u,\,t)\zeta_2(v,\,-t)\,\mathrm{d}t. \qquad (10.09)$$

Inserting in the boundary conditions the expressions (10.05), (10.06) and (10.09) for P', Q', Q'^* and putting $v=v_0$, we obtain for the unknown functions $p(t)$ and $q(t)$ the equations

$$\left.\begin{aligned} p(t)\cdot 2\zeta_2'+q(t)\zeta_2 &= -2\zeta_1' \\ p(t)\zeta_2+q(t)\,(2\zeta_2'-2t\zeta_2) &= -\,\zeta_1, \end{aligned}\right\} \qquad (10.10)$$

where we have put for brevity

$$\zeta_1 = \zeta_1(v_0,\,-t); \quad \zeta_1' = \left(\frac{\partial\zeta_1(v,\,-t)}{\partial v}\right)_{v=v_0} \qquad (10.11)$$

and similarly for ζ_2 and ζ_2'.

Solution of equations (10.10) yields

$$p(t) = \frac{-\zeta_1\zeta_2+4\zeta_1'\zeta_2'-4t\zeta_1'\zeta_2}{\zeta_2^2-4\zeta_2'^2+4t\zeta_2'\zeta_2}, \qquad (10.12)$$

$$q(t) = \frac{2\zeta_1\zeta_2'-2\zeta_2\zeta_1'}{\zeta_2^2-4\zeta_2'^2+4t\zeta_2'\zeta_2}. \qquad (10.13)$$

We note that the numerator of the fraction for $q(t)$ is the Wronskian, which is equal to

$$2\zeta_1\zeta_2'-2\zeta_2\zeta_1' = -\frac{2i}{v_0}\,\mathrm{e}^{\frac{\pi}{2}t}. \qquad (10.14)$$

Introducing into (10.05) and (10.06) the values of $p(t)$ and $q(t)$ just found we obtain the potentials of the reflected wave and, therefore, the solution of our problem.

11. *Representation of the Solution in the Form of Series*

The potentials of the reflected wave can be also represented in series form. As in the case of the primary field considered in Section 4, these series can be obtained either as a sum of residues at the poles $t = -(2n+1)i$ in the lower half-plane, or as a sum of residues at the poles $t = +(2n+1)i$ in the upper half-plane. When calculating the residues, it is to be borne in mind that the common denominator of the functions $p(t)$ and $q(t)$ has a simple root in the point $t = -i$ and has no other roots. Therefore, the pole $t = -i$ is multiple (double), while the other poles are simple.

Let us first find the residue at the double pole $t = -i$.

From formulae (2.11)–(2.15) it is easy to deduce that in the vicinity of $t = -i$ we will have

$$\zeta_1(u, t) = e^{i\frac{u}{2}}\left\{1+\left(-\frac{\pi}{4}+\frac{i}{2}\log u\right)(t+i)+\ldots\right\}, \tag{11.01}$$

$$\zeta_2(v, -t) = e^{-i\frac{v}{2}}\left\{1+\left(\frac{\pi}{4}+\frac{i}{2}\log v\right)(t+i)+\ldots\right\}, \tag{11.02}$$

whence

$$\zeta_1(u, t)\zeta_2(v, -t) = e^{i\frac{u-v}{2}}\left\{1+\frac{i}{2}(t+i)\log(uv)+\ldots\right\}. \tag{11.03}$$

The common denominator of the functions $p(t)$ and $q(t)$ can be written as

$$(\zeta_2-2i\zeta_2')^2+4(t+i)\zeta_2'\zeta_2 = -2i\, e^{-iv_0}(t+i)\times$$
$$\times\left\{1+(t+i)\left(\frac{\pi}{2}+i\log v_0-\frac{1}{v_0}+\frac{i}{2v_0^2}\right)+\ldots\right\}. \tag{11.04}$$

The values of the numerators of $p(t)$ and $q(t)$ for $t = -i$ are respectively

$$(2\zeta_1'+i\zeta_1)(2\zeta_2'+i\zeta_2)-2i(\zeta_1\zeta_2'-\zeta_2\zeta_1') = \frac{2i}{v_0}, \tag{11.05}$$

$$2(\zeta_1\zeta_2'-\zeta_2\zeta_1') = -\frac{2}{v_0}. \tag{11.06}$$

In the vicinity of $t = -i$ we have therefore

$$p(t) = -\frac{1}{t+i}\cdot\frac{e^{iv_0}}{v_0}+p_{00}+\ldots \tag{11.07}$$

$$q(t) = -\frac{i}{t+i}\cdot\frac{e^{iv_0}}{v_0}+q_{00}+\ldots, \tag{11.08}$$

where p_{00} and q_{00} are constants that need not be determined here, since they drop out from the expressions for the field. Inserting the expressions (11.03), (11.07) and (11.08) into (10.05) and (10.06), we obtain the values of the residues at the point $t = -i$, namely

$$P_{00}' = 2iCp_{00}\, e^{i\frac{u-v}{2}}+\frac{C}{v_0}e^{iv_0}\cdot e^{i\frac{u-v}{2}}\log(uv), \tag{11.09}$$

$$Q_{00}' = 2iCq_{00}\, e^{i\frac{u-v}{2}}+\frac{iC}{v_0}e^{iv_0}\cdot e^{i\frac{u-v}{2}}\log(uv). \tag{11.10}$$

These expressions contain logarithmic terms and do not remain finite on the axis of the paraboloid. The quantities S, T, F, and G_1, however, calculated from the above expressions by means of the formulae (9.12)–(9.15), are free from logarithmic terms. The contributions to the field arising from the terms P'_{00} and Q'_{00} in the potentials are therefore finite. For the electric field we have, e.g.,

$$\left.\begin{aligned} 2u(E'_u)_0 &= \frac{C}{v_0} e^{iv_0} \cdot e^{i\frac{u-v}{2}} \sqrt{(uv)} \cos\varphi, \\ 2v(E'_v)_0 &= \frac{C}{v_0} e^{iv_0} \cdot e^{i\frac{u-v}{2}} \sqrt{(uv)} \cos\varphi, \\ (E_\varphi)_0 &= -\frac{C}{v_0} e^{iv_0} \cdot e^{i\frac{u-v}{2}} \sqrt{(uv)} \sin\varphi, \end{aligned}\right\} \tag{11.11}$$

This corresponds to a plane wave polarized in the x-direction.

The calculation of the residues at the poles $t = -(2n+1)i$, where $n=1, 2, \ldots$ meet with no difficulties whatever. Using the formulae (3.01) and (3.02), we obtain for the potentials of the reflected wave the following series

$$P' = P'_{00} + C \sum_{n=1}^{\infty} p_n^{(1)} \xi_n(u) \xi_n(v), \tag{11.12}$$

$$Q' = Q'_{00} + C \sum_{n=1}^{\infty} q_n^{(1)} \xi_n(u) \xi_n(v), \tag{11.13}$$

where

$$p_n^{(1)} = 2i(-1)^n (n!)^2 p[-(2n+1)i], \tag{11.14}$$

$$q_n^{(1)} = 2i(-1)^n (n!)^2 q[-(2n+1)i]. \tag{11.15}$$

Calculation of the quantities p_n, q_n yields

$$p_n^{(1)} = -2i \cdot \frac{\xi_n \bar{\eta}_n - 4\xi'_n \bar{\eta}'_n - 4i(2n+1)\xi_n \bar{\eta}'_n}{\xi_n^2 - 4\xi_n'^2 - 4i(2n+1)\xi_n \xi'_n}, \tag{11.16}$$

$$q_n^{(1)} = -\frac{4i}{v_0} \cdot \frac{1}{\xi_n^2 - 4\xi_n'^2 - 4i(2n+1)\xi_n \xi'_n}, \tag{11.17}$$

where the functions ξ_n, η_n have v_0 as argument. We note that the denominator of the fractions p_n, q_n can be written as

$$\begin{aligned} &\xi_n^2 - 4\xi_n'^2 - 4i(2n+1)\xi_n \xi'_n \\ &= \frac{4}{v_0} n(n+1) \left(n\xi_{n-1,1}^2 - (n+1)\xi_{n,1}^2\right). \end{aligned} \tag{11.18}$$

Let us now find the expressions for the potentials P', Q' as a sum of residues at the poles $t = +(2n+1)i$. All the poles in the upper half-plane are simple poles, and we have

$$P' = C \sum_{n=0}^{\infty} p_n^{(2)} \bar{\eta}_n(u)\eta_n(v), \tag{11.19}$$

$$Q' = C \sum_{n=0}^{\infty} q_n^{(2)} \bar{\eta}_n(u)\eta_n(v), \tag{11.20}$$

where

$$p_n^{(2)} = \frac{2i(-1)^n}{(n!)^2} p[(2n+1)i] \tag{11.21}$$

$$q_n^{(2)} = \frac{2i(-1)^n}{(n!)^2} q[(2n+1)i]. \tag{11.22}$$

Expressing the quantities $p_n^{(2)}$, $q_n^{(2)}$ in terms of the functions $\xi_n(v_0)$, $\eta_n(v_0)$, we obtain

$$p_n^{(2)} = -2i\,\frac{\xi_n\eta_n - 4\xi_n'\eta_n' + 4i(2n+1)\xi_n'\eta_n}{\eta_n^2 - 4\eta_n'^2 + 4i(2n+1)\eta_n'\eta_n}. \tag{11.23}$$

$$q_n^{(2)} = \frac{4i}{v_0} \cdot \frac{1}{\eta_n^2 - 4\eta_n'^2 + 4i(2n+1)\eta_n'\eta_n}. \tag{11.24}$$

In particular, we will have for $n=0$

$$p_0^{(2)} = v_0\, e^{iv_0}; \quad q_0^{(2)} = -iv_0\, e^{iv_0}. \tag{11.25}$$

We calculate the field that corresponds to the zero terms of the expansions (11.19) and (11.20), that is, to the potentials

$$P_0^{(2)} = Cv_0\, e^{iv_0} \bar{\eta}_0(v), \tag{11.26}$$

$$Q_0^{(2)} = -iCv_0\, e^{iv_0} \bar{\eta}_0(u)\eta_0(v). \tag{11.27}$$

We have

$$S_0^{(2)} = 0; \quad T_0^{(2)} = 0, \tag{11.28}$$

$$F_{10}^{(2)} = \frac{Cv_0}{uv}\, e^{i\left(v_0 + \frac{u-v}{2}\right)}, \tag{11.29}$$

$$G_{10}^{(2)} = -\frac{iCv_0}{uv}\, e^{i\left(v_0 + \frac{u-v}{2}\right)}. \tag{11.30}$$

The parabolic components of the field are obtained by substituting the quantities (11.28)–(11.30) into (9.16) and (9.17). This field corresponds to

electric potentials

$$A_x^{(2)} = 0; \quad A_y^{(2)} = 0, \tag{11.31}$$

$$A_z^{(2)} = A_0^{(2)} = -C\frac{ax}{r^2}e^{ik(a+z)}. \tag{11.32}$$

These expressions can be regarded as approximations to the true field only if the conditions $u \gg 1$, $uv \gg v_0^2$ are satisfied; in the general case further members of the series must be taken into account.

The series for the auxiliary functions S, T, F_1, G_1 are obtained from the series for P', Q' by means of the formulae deduced in Section 8. We put

$$a_n^{(1)} = (n+1)(p_n^{(1)}+iq_n^{(1)}) \qquad (n = 1, 2, \ldots), \tag{11.33}$$

$$a_0^{(1)} = 2i(p_{00}+iq_{00}), \tag{11.34}$$

$$b_n^{(1)} = n(p_n^{(1)}-iq_n^{(1)}) \qquad (n = 1, 2, \ldots), \tag{11.35}$$

$$b_0^{(1)} = \frac{2}{v_0}e^{iv_0}. \tag{11.36}$$

We will then have

$$S'+iT' = iC\sum_{n=0}^{\infty} a_n^{(1)}\xi_n(u)\bar{\xi}_n(v), \tag{11.37}$$

$$S'-iT' = iC\sum_{n=0}^{\infty} b_n^{(1)}\xi_n(u)\bar{\xi}_n(v), \tag{11.38}$$

$$F_1'+iG_1' = \frac{C}{\sqrt{(uv)}}\sum_{n=0}^{\infty}(n+1)a_n^{(1)}\xi_{n1}(u)\bar{\xi}_{n1}(v), \tag{11.39}$$

$$F_1'-iG = \frac{C}{\sqrt{(uv)}}\sum_{n=0}^{\infty}(n+1)b_{n+1}^{(1)}\xi_{n1}(u)\bar{\xi}_{n1}(v). \tag{11.40}$$

In a similar way series proceeding in functions $\bar{\eta}_n(u)\eta_n(v)$ can be obtained. We put

$$a_n^{(2)} = n(p_n^{(2)}+iq_n^{(2)}), \quad a_0^{(2)} = 0; \tag{11.41}$$

$$b_n^{(2)} = (n+1)(p_n^{(2)}-iq_n^{(2)}), \quad b_0^{(2)} = 0 \tag{11.42}$$

and we obtain

$$S'+iT' = -iC\sum_{n=0}^{\infty} a_n^{(2)}\bar{\eta}_n(u)\eta_n(v), \tag{11.43}$$

$$S'-iT' = -iC\sum_{n=0}^{\infty} b_n^{(2)}\bar{\eta}_n(u)\eta_n(v), \tag{11.44}$$

$$F_1'+iG_1' = \frac{2Cv_0}{uv} e^{iv_0} \cdot e^{i\frac{u-v}{2}} +$$

$$+\frac{C}{\sqrt{(uv)}} \sum_{n=0}^{\infty} (n+1)a_{n+1}^{(2)}\bar{\eta}_{n1}(u)\eta_{n1}(v), \tag{11.45}$$

$$F_1'-iG_1' = \frac{C}{\sqrt{(uv)}} \sum_{n=0}^{\infty} (n+1)b_n^{(2)}\bar{\eta}_{n1}(u)\eta_{n1}(v). \tag{11.46}$$

Substitution of these expressions into (9.16) and (9.17) gives the field of the reflected wave.

It remains to investigate the convergence of the series obtained. This is easily done with the help of the asymptotic expressions for the functions ξ_n and η_n given in Section 3. These expressions are

$$\xi_n(u) = J_0\left((1-i)\sqrt{\{(2n+1)u\}}\right), \tag{11.47}$$

$$\eta_n(u) = i\pi H_0^{(2)}\left((1-i)\sqrt{\{(2n+1)u\}}\right). \tag{11.48}$$

Using them, we find for the coefficients of the later terms of our series the approximate values

$$p_n^{(1)} = -i4\pi\, e^{-2w+2iw}, \tag{11.49}$$

$$q_n^{(1)} = \frac{4\pi}{2n+1} e^{-2w+2iw}, \tag{11.50}$$

$$p_n^{(2)} = -\frac{i}{\pi} e^{2w+2iw}; \tag{11.51}$$

$$q_n^{(2)} = -\frac{i}{\pi(2n+1)} e^{2w+2iw}, \tag{11.52}$$

where we have put for brevity

$$w = \sqrt{\{(2n+1)v_0\}}. \tag{11.53}$$

It is seen that the condition of convergence of the series (11.12), (11.13), and of other series proceeding in functions $\xi_n(u)\bar{\xi}_n(v)$, is

$$\sqrt{u}+\sqrt{v} < 2\sqrt{v_0}, \tag{11.54}$$

while the condition of convergence of (11.19), (11.20) and of other series proceeding in functions $\bar{\eta}_n(u)\eta_n(v)$ is

$$\sqrt{u}+\sqrt{v} > 2\sqrt{v_0}. \tag{11.55}$$

The boundary between the two regions of convergence is a surface defined

by the equation

$$R+r = 2a. \tag{11.56}$$

This is a surface of revolution whose section by the plane of symmetry is a parabola; the axis of this parabola is at right angles to the axis of the given paraboloid, and the vertex of the parabola lies at the point $z=0$, $r=a$, that is, on the intersection of the paraboloid with its focal plane.

12. *The Field in the Wave Zone*

When the three numbers u, v, v_0 are large compared with unity, expressions for the field can be obtained that correspond to the approximation of geometrical optics. Our starting point will be the integral representation (10.05), (10.06) of the potentials P' and Q'. In the case considered, the relevant section of the integration path in these integrals corresponds to finite values of t. But for finite t we can apply the asymptotic expressions (2.11) and (2.12) for the functions ζ_1 and ζ_2. Using these expressions, we obtain for $q(t)$ and $p(t)$

$$q(t) = -\, iv_0^{-it}\, \mathrm{e}^{iv_0} \cdot \left(\frac{1}{t+i}+\frac{it+1}{2v_0}+\ldots\right), \tag{12.01}$$

$$p(t) = q(t)\cdot(t+\ldots), \tag{12.02}$$

where the rows of dots denote quantities of the order $1/v_0^2$. With the same accuracy we have

$$\zeta_1(u,\, t)\zeta_2(v,\, -t) = (uv)^{-\frac{1}{2}+\frac{it}{2}} \cdot \mathrm{e}^{i\frac{u-v}{2}} \times \\ \times\left(1-\frac{i}{4}(t+i)^2\frac{u-v}{uv}+\ldots\right). \tag{12.03}$$

Inserting these expressions in the integral

$$Q' = \frac{iC}{2}\int\limits_{-\infty}^{+\infty}\frac{q(t)}{\cosh\dfrac{\pi t}{2}}\,\zeta_1(u,\, t)\zeta_2(v,\, -t)\,\mathrm{d}t, \tag{12.04}$$

we can write it in the form

$$Q' = \frac{C\mathrm{e}^{i\left(v_0+\frac{u-v}{2}\right)}}{2\sqrt{(uv)}}\int\limits_{-\infty}^{+\infty}\frac{\mathrm{d}t}{\cosh\dfrac{\pi t}{2}}\left(\frac{v_0^2}{uv}\right)^{-\frac{it}{2}}\times \\ \times\left\{\frac{1}{t+i}+\frac{1-it}{4}\cdot\frac{u-v}{uv}+\frac{1+it}{2v_0}+\ldots\right\}. \tag{12.05}$$

This integral can be evaluated without further approximations. Using the relations

$$\int_{-\infty}^{+\infty} \frac{z^{-it}\,\mathrm{d}t}{(t+i)\cosh\frac{\pi t}{2}} = -\frac{2i}{z}\log(1+z^2), \tag{12.06}$$

$$\int_{-\infty}^{+\infty} \frac{(1-it)}{\cosh\frac{\pi t}{2}}\cdot z^{-it}\,\mathrm{d}t = \frac{8z}{(1+z^2)^2}, \tag{12.07}$$

$$\int_{-\infty}^{+\infty} \frac{(1+it)}{\cosh\frac{\pi t}{2}}\cdot z^{-it}\,\mathrm{d}t = \frac{8z^3}{(1+z^2)^2}. \tag{12.08}$$

we obtain the following approximate expression for Q':

$$Q' = Ce^{i\left(v_0+\frac{u-v}{2}\right)}\left\{-\frac{i}{v_0}\log\left(1+\frac{v_0^2}{uv}\right)+\frac{v_0(u-v+2v_0)}{(v_0^2+uv)^2}\right\}. \tag{12.09}$$

The integral P' can be evaluated in a similar way, but we can avoid integrations if we use the relation $p(t)=tq(t)$, valid, according to (12.02), in our approximation. From this relation it follows that $P'=-2Q'^*$, where Q'^* is the quantity (10.09). Calculating P' by means of the formula

$$P' = -2L_uQ', \tag{12.10}$$

we obtain

$$P' = Ce^{i\left(v_0+\frac{u-v}{2}\right)}\left\{-\frac{1}{v_0}\log\left(1+\frac{v_0^2}{uv}\right)+\frac{2v_0}{v_0^2+uv}+\right.$$
$$\left.+\frac{iv_0}{(v_0^2+uv)^3}\left[(u-v)(uv-3v_0^2)-2v_0^3+6v_0uv\right]\right\}. \tag{12.11}$$

The auxiliary functions S', T', F', G_1' are obtained from (12.09) and (12.11) by means of differentiations. We have

$$S' = Ce^{i\left(v_0+\frac{u-v}{2}\right)}\left\{\frac{2iv_0^3}{(v_0^2+uv)^2}+\frac{4v_0^3(u-v)(v_0^2-2uv)}{(v_0^2+uv)^4}+\right.$$
$$\left.+\frac{8v_0^2uv(uv-2v_0^2)}{(v_0^2+uv)^4}\right\}, \tag{12.12}$$

$$T' = 0, \tag{12.13}$$

$$F_1' = iG_1' = Ce^{i\left(v_0+\frac{u-v}{2}\right)}\left\{\frac{v_0}{v_0^2+uv}-\frac{2iv_0^3(u-v)}{(v_0^2+uv)^3}+\frac{4iv_0^2uv}{(v_0^2+uv)^3}\right\}. \tag{12.14}$$

The difference $F_1'-iG'$ will be of higher order than F_1', namely

$$F_1'-iG_1' = -Ce^{i\left(v_0+\frac{u-v}{2}\right)}\frac{4v_0^3}{(v_0^2+uv)^3} \sim 0. \tag{12.15}$$

Substitution of the obtained values of the auxiliary functions into (9.16) and (9.17) yields the parabolic field components for the reflected wave. Adding them to those given by (9.04) and (9.05) for the primary field, we obtain the total field. Because of the complicated character of the formulae for the parabolic components of the field we do not write them out, but only verify the fulfilment of the boundary conditions. It follows from the conditions $E_\varphi=0$; $H_v=0$ that on the surface of the paraboloid $v=v_0$ we must have

$$F_1' = T^\circ - F_1^\circ, \tag{12.16}$$

$$G_1' = -2\frac{\partial T^\circ}{\partial u}, \tag{12.17}$$

where T° and F_1°, belong to the primary field and are given by (9.20)–(9.23). In deducing (12.16) and (12.17) we have used the relations $G_1^\circ=0$ and $T'=0$. For $v=v_0$ we have

$$F_1' = iG_1' = Ce^{i\frac{u+v_0}{2}}\left(\frac{1}{u+v_0}+\frac{2i}{(u+v_0)^2}\right). \tag{12.18}$$

Comparing (12.18) with the difference of the expressions (9.21) and (9.22) for T° and F_1°, we see that the terms of the orders $1/(u+v_0)$ and $1/(u+v_0)^2$ in equation (12.16) are the same on both sides. To this approximation, both sides of the equation (12.17) also agree.

In conclusion we note that, although the expressions found in this section for S', T', F_1', G_1', are proved for large distances from the axis (the quantities u, v, were supposed to be large) they are valid without this restriction. The expressions concerned remain holomorphic functions of the Cartesian coordinates x, y, z even near the axis and can be used in all cases when the correction terms are small compared with the main terms, including the region near and on the axis.

13. *Rectangular Field Components for the Reflected Wave*

For the calculation of the rectangular field components it is expedient to use the formulae (7.21) and (7.22) for the electric and the magnetic potential. We begin by stating the complete formulae, and then we go over to the approximation considered in the previous section.

According to (7.21), the rectangular components of the electric potentials are equal to

$$A_x = iT; \quad A_y = 0; \quad A_z = -ikxG_1; \tag{13.01}$$

$$\operatorname{div} \mathbf{A} = ikA_0 = -ik^2xF_1 \cdot \tag{13.02}$$

On the other hand, according to (7.22) the magnetic potentials are equal to

$$B_x = 0; \quad B_y = S; \quad B_z = -kyF_1; \tag{13.03}$$

$$\operatorname{div} \mathbf{B} = ikB_0 = k^2yG_1. \tag{13.04}$$

The field can be, therefore, expressed in two forms, namely

$$\left.\begin{aligned} E_x &= \frac{\partial(kxF_1)}{\partial x} - kT = -\frac{\partial(kyF_1)}{\partial y} - \frac{\partial S}{\partial z} \\ E_y &= \frac{\partial(kxF_1)}{\partial y} = \frac{\partial(kyF_1)}{\partial x} \\ E_z &= \frac{\partial(kxF_1)}{\partial z} + k^2xG_1 = \frac{\partial S}{\partial x} \end{aligned}\right\} \tag{13.05}$$

and

$$\left.\begin{aligned} H_x &= -i\frac{\partial(kxG_1)}{\partial y} = -i\frac{\partial(kyG_1)}{\partial x}, \\ H_y &= i\frac{\partial T}{\partial z} + i\frac{\partial(kxG_1)}{\partial x} = -i\frac{\partial(kyG_1)}{\partial y} - ikS, \\ H_z &= -i\frac{\partial T}{\partial y} = -i\frac{\partial(kyG_1)}{\partial z} + ik^2yF_1. \end{aligned}\right\} \tag{13.06}$$

Turning now to the approximate formulae for the reflected wave, we can put $T'=0$, $F_1'=iG_1'$. The field of the reflected wave will, therefore, correspond approximately to a vector-potential with a single non-vanishing component A_z', which is equal to

$$A_z' = Ce^{ik(a+z)} \cdot x\left\{-\frac{a}{a^2+r^2} - \frac{4ia^2r^2}{k(a^2+r^2)^3} + \frac{4ia^3z}{k^2(a^2+r^2)^3}\right\}. \tag{13.07}$$

according to (13.01) and (12.14).

The field components in the directions of the x- and of the y-axis (at right angles to the axis of symmetry) can be written as

$$E_x' = H_y' = \frac{k}{2r}\frac{\partial(r^2F_1)}{\partial r} + \cos 2\varphi \cdot \frac{k}{2}r\frac{\partial F_1}{\partial r}, \tag{13.08}$$

$$E_y' = -H_x' = \sin 2\varphi \cdot \frac{k}{2}r\frac{\partial F_1}{\partial r}, \tag{13.09}$$

where

$$\frac{k}{2r}\frac{\partial(r^2F_1)}{\partial r} = Ce^{ik(a+z)}\left\{\frac{a^3}{(a^2+r^2)^2}+\right.$$
$$\left.+\frac{4ia^2[r^2(2a^2-r^2)+az(2r^2-a^2)]}{k(a^2+r^2)^4}\right\}, \tag{13.10}$$

$$\frac{kr}{2}\frac{\partial F_1}{\partial r} = Ce^{ik(a+z)}\left\{-\frac{ar^2}{(a^2+r^2)^2}-\frac{4ia^2r^2(2r^2-a^2-3az)}{k(a^2+r^2)^4}\right\}. \tag{13.11}$$

The components in the direction of the z-axis (the axis of symmetry) can be calculated with help of the approximate expressions

$$S' = Ce^{ik(a+z)}\cdot\frac{2ia^3}{k(a^2+r^2)^2}; \qquad T' = 0. \tag{13.12}$$

We obtain

$$E_z' = -Ce^{ik(a+z)}\cdot\frac{8ia^3r\cos\varphi}{k(a^2+r^2)^3}; \qquad H_z' = 0. \tag{13.13}$$

The main terms in the field of the reflected wave have, in Cartesian coordinates, the following explicit expressions

$$\left.\begin{aligned} E_x' &= C\frac{a(a^2+y^2-x^2)}{(a^2+x^2+y^2)^2}e^{ik(a+z)},\\ E_y' &= -C\cdot\frac{2axy}{(a^2+x^2+y^2)^2}e^{ik(a+z)},\\ E_z' &= -C\cdot\frac{8ia^3x}{k(a^2+x^2+y^2)^3}e^{ik(a+z)},\end{aligned}\right\} \tag{13.14}$$

$$\left.\begin{aligned} H_x' &= C\frac{2axy}{(a^2+x^2+y^2)^2}e^{ik(a+z)},\\ H_y' &= C\frac{a(a^2+y^2-x^2)}{(a^2+x^2+y^2)^2}e^{ik(a+z)},\\ H_z' &= 0.\end{aligned}\right\} \tag{13.15}$$

These expressions satisfy the equations $\operatorname{div}\mathbf{E}'=0$; $\operatorname{div}\mathbf{H}'=0$ rigorously; the remaining Maxwell equations are satisfied only approximately. Together with the primary field, the above expressions satisfy approximately the boundary conditions.

CHAPTER 4

DIFFRACTION OF A PLANE ELECTROMAGNETIC WAVE BY A PERFECTLY CONDUCTING PARABOLOID OF REVOLUTION†

Abstract—Diffraction of a plane electromagnetic wave of arbitrary polarization, incident on the outer surface of a perfectly conducting paraboloid of revolution at an arbitrary angle is analysed. An exact solution of the problem has been obtained as well as asymptotic formulae for the current distribution on the surface, which are valid under the condition that the wavelength is small in comparison with the focal length of the paraboloid.

INTRODUCTION

In Chapter 2 are presented the results of the solution of the problem of diffraction of a plane electromagnetic wave by a conducting paraboloid of revolution. These results are of general importance since they give the current distribution on convex conducting bodies.

The formulae in Chapter 2 are given without a detailed derivation, which, however, would be of interest from both the viewpoint of diffraction theory and from a mathematical viewpoint. The derivation of the exact solution‡ is facilitated by using the potentials introduced in Chapter 3. However, essential parts of the derivation are the method of transforming the diffraction series into a contour integral and the approximate evaluation of the integral (the asymptotic summation of the series). These parts of the derivation, which were omitted completely in Chapter 2. are considered in Sections 2 and 3.§ Moreover, the formulae were obtained only for one definite direction of polarization of the incident wave, which

† Fock and Fedorov, 1958.

‡ The new derivation of the exact expressions for the field (on the basis of the potentials introduced by Fock in Chapter 3) is due to Fedorov (Sects. 1 and 4, herein).

§ Sections 2 and 3 are only a modification of the Fock calculations, which he used but did not explain in Chapter 2.

we shall designate as the first type of polarization.† In this paper, a second type of polarization is also analysed, hence, the results of Chapter 2 are generalized to the case of arbitrary incident wave polarization.‡

1. *Exact Solution of the Problem for the First Type of Plane Wave Polarization*

Let the equation of the paraboloid (Fig. 1) be

$$x^2+y^2-2az-a^2 = 0 \tag{1.01}$$

We take the XOZ plane as the plane of incidence and call the polarization of a plane wave whose magnetic field is directed along the y-axis the first type of polarization.

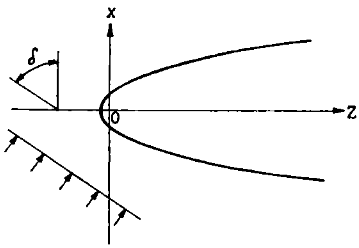

FIG. 1.

The Cartesian components of the electromagnetic field are

$$\left.\begin{aligned} E_x &= E_0 \cos\delta\, e^{i\Omega} & H_x &= 0, \\ E_y &= 0 & H_y &= E_0 e^{i\Omega}, \\ E_z &= -E_0 \sin\delta\, e^{i\Omega} & H_z &= 0, \end{aligned}\right\} \tag{1.02}$$

where $\Omega = k(x \sin\delta + z \cos\delta)$ is the phase of the plane wave; $k = 2\pi/\lambda = \omega/c$ is the wave number in a vacuum; the time dependence is given by the factor $e^{-i\omega t}$.

If the following parabolic coordinates are introduced

$$\left.\begin{aligned} u = k(R+z); \quad v = k(R-z); \quad \varphi = \arctan\frac{y}{x}, \\ R = \sqrt{(x^2+y^2+z^2)}, \end{aligned}\right\} \tag{1.03}$$

† The first type of polarization corresponds to the case where the magnetic vector of the incident wave is tangential to the surface on the shadow boundary. For the second type of polarization, the electric vector is tangential.

‡ The calculations for the second type of polarization in Section 4 were carried out by A. A. Fedorov.

then the equation of the paraboloid is written as

$$v = v_0 = ka. \tag{1.04}$$

The covariant field components in parabolic coordinates will be for the field (1.02):

$$\left.\begin{aligned}
E_u &= \frac{E_0}{2k} e^{i\Omega}\left(\sqrt{\left(\frac{v}{u}\right)}\cos\delta\cos\varphi - \sin\delta\right), & H_u &= \frac{E_0}{2k}\sqrt{\left(\frac{v}{u}\right)} e^{i\Omega}\sin\varphi,\\
E_v &= \frac{E_0}{2k} e^{i\Omega}\left(\sqrt{\left(\frac{u}{v}\right)}\cos\delta\cos\varphi + \sin\delta\right), & H_v &= \frac{E_0}{2k}\sqrt{\left(\frac{u}{v}\right)} e^{i\Omega}\sin\varphi,\\
E_\varphi &= -\frac{E_0}{k}\sqrt{(uv)}\, e^{i\Omega}\cos\delta\sin\varphi, & H_\varphi &= \frac{E_0}{k}\sqrt{(uv)}\, e^{i\Omega}\cos\varphi,
\end{aligned}\right\} \tag{1.05}$$

where, in the new coordinates,

$$\Omega = \frac{1}{2}(u-v)\cos\delta + \sqrt{(uv)}\sin\delta\cos\varphi. \tag{1.06}$$

Since the components E_u, E_v, H_φ are even and the H_u, H_v, E_φ are odd functions of φ, we expand them in the Fourier series

$$\left.\begin{aligned}
E_u &= \frac{1}{2}E_u^{(0)} + \sum_{s=1}^{\infty} E_u^{(s)}\cos s\varphi, & E_v &= \ldots, & H_\varphi &= \ldots\\
H_u &= \sum_{s=1}^{\infty} H_u^{(s)}\sin s\varphi, & H_v &= \ldots, & E_\varphi &= \ldots
\end{aligned}\right\} \tag{1.07}$$

where in the first line the rows of dots denote cosine series and in the second line sine series.

Knowing the series expansion of $e^{i\Omega}$ (see (5.03) of Chapter 3)

$$e^{i\Omega} = e^{\frac{i}{2}(u-v)\cos\delta}\left\{J_0\left(\sqrt{(uv)}\sin\delta\right) + 2\sum_{s=1}^{\infty} i^s J_s\left(\sqrt{(uv)}\sin\delta\right)\cos s\varphi,\right\} \tag{1.08}$$

we can write the Fourier coefficients of the field (1.05) in the following form which is convenient for further transformation

$$\left.\begin{aligned}
E_u^{(s)} &= -\frac{E_0 i^s}{k\sin\delta}\left[1 + 2i\cos\delta\frac{\partial}{\partial u}\right] e^{\frac{i}{2}(u-v)\cos\delta} J_s\left(\sqrt{(uv)}\sin\delta\right),\\
E_v^{(s)} &= \frac{E_0 i^s}{k\sin\delta}\left[1 - 2i\cos\delta\frac{\partial}{\partial v}\right] e^{\frac{i}{2}(u-v)\cos\delta} J_s\left(\sqrt{(uv)}\sin\delta\right),
\end{aligned}\right\} \tag{1.09}$$

$$\left.\begin{aligned}
E_\varphi^{(s)} &= \frac{2E_0 i^{s+1}}{k} s \cot\delta\, e^{\frac{i}{2}(u-v)\cos\delta} J_s(\sqrt{(uv)}\sin\delta),\\
H_u^{(s)} &= -\frac{E_0}{ku}\frac{i^{s+1}}{\sin\delta} s\, e^{\frac{i}{2}(u-v)\cos\delta} J_s(\sqrt{(uv)}\sin\delta),\\
H_v^{(s)} &= -\frac{E_0}{kv}\frac{i^{s+1}}{\sin\delta} s\, e^{\frac{i}{2}(u-v)\cos\delta} J_s(\sqrt{(uv)}\sin\delta),\\
H_\varphi^{(s)} &= \frac{2E_0 i^s v}{k\sin\delta}\left[\cos\delta - 2i\frac{\partial}{\partial v}\right] e^{\frac{i}{2}(u-v)\cos\delta} J_s(\sqrt{(uv)}\sin\delta).
\end{aligned}\right\} \quad (1.09)$$

On the other hand, the components $E_u^{(s)}$, $E_v^{(s)}$, $\cdots H_\varphi^{(s)}$ can be expressed in terms of the auxiliary functions C_{s-1}, D_{s-1}, F_s, G_s by means of formulae (7.05) and (7.06) of Chapter 3.

Comparing our expressions (1.09) with these formulae, we find the form of the auxiliary functions for the plane wave

$$\left.\begin{aligned}
C_{s-1}^0 &= \frac{2E_0 i^s}{k}\sqrt{(uv)}^{-s+1} e^{\frac{i}{2}(u-v)\cos\delta} J_{s-1}\{\sqrt{(uv)}\sin\delta\},\\
D_{s-1}^0 &= 0,\\
F_s^0 &= \frac{2E_0 i^{s-1}}{k}\sqrt{(uv)}^{-s}\cot\delta\, e^{\frac{i}{2}(u-v)\cos\delta} J_s\{\sqrt{(uv)}\sin\delta\},\\
G_s^0 &= -\frac{2E_0 i^s}{k\sin\delta}\sqrt{(uv)}^{-s} e^{\frac{i}{2}(u-v)\cos\delta} J_s\{\sqrt{(uv)}\sin\delta\}.
\end{aligned}\right\} \quad (1.10)$$

Knowing the functions (1.10), we can find for the plane wave the parabolic potentials P_{s-1}^0 and Q_{s-1}^0, introduced in Chapter 3. These potentials can be written as integrals (see formula (5.05) of Chapter 3)

$$\left.\begin{aligned}
P_{s-1}^0 &= \frac{E_0 i^s}{\pi k\sin\delta}\sqrt{(uv)}^{-s+1}\int_{-\infty}^{+\infty} p(t)\psi(u, s-1, t)\psi(v, s-1, -t)\,\mathrm{d}t,\\
Q_{s-1}^0 &= \frac{E_0 i^s}{\pi k\sin\delta}\sqrt{(uv)}^{-s+1}\int_{-\infty}^{+\infty} q(t)\psi(u, s-1, t)\psi(v, s-1, -t)\,\mathrm{d}t,
\end{aligned}\right\} \quad (1.11)$$

where $p(t)$, $q(t)$ are unknown functions and $\psi(u, s-1, t)$, $\psi(v, s-1, -t)$ are functions introduced by the formula (2.23) of Chapter 3.

Using the relation (5.11) of Chapter 3, the first formula (1.10) can be rewritten in the form

$$C_{s-1}^0 = \frac{E_0}{\pi k\sin\delta}\sqrt{(uv)}^{-s+1}\int_{-\infty}^{+\infty}\psi(u, s-1, t)\psi(v, s-1, -t)\left(\tan\frac{\delta}{2}\right)^{it}\mathrm{d}t. \quad (1.12)$$

Substituting (1.11) and (1.12) into (7.01) and (7.02) of Chapter 3 (these formulae relate the functions C_{s-1} and D_{s-1} to the parabolic

potentials P_{s-1} and Q_{s-1}) and solving the resulting equations, we find

$$p(t) = -\frac{2t}{s^2+t^2}\left(\tan\frac{\delta}{2}\right)^{it}, \quad q(t) = -\frac{2s}{s^2+t^2}\left(\tan\frac{\delta}{2}\right)^{it} \tag{1.13}$$

For the potentials P'_{s-1} and Q'_{s-1} of the secondary (reflected) wave we assume the following integral representations

$$\left.\begin{aligned} P'_{s-1} &= \frac{2E_0 i^s}{\pi k \sin\delta}\sqrt{(uv)^{-s+1}}\int_{-\infty}^{+\infty}\frac{t}{s^2+t^2}\left(\tan\frac{\delta}{2}\right)^{it}\times \\ &\quad \times p^1(t)\psi(u, s-1, t)\zeta_1(v, s-1, -t)\,\mathrm{d}t, \\ Q'_{s-1} &= \frac{2E_0 i^s}{\pi k \sin\delta}\sqrt{(uv)^{-s+1}}\int_{-\infty}^{+\infty}\frac{s}{s^2+t^2}\left(\tan\frac{\delta}{2}\right)^{it}\times \\ &\quad \times q^1(t)\psi(u, s-1, t)\zeta_1(v, s-1, -t)\,\mathrm{d}t, \end{aligned}\right\} \tag{1.14}$$

The function ζ_1 has been introduced into the expansion (1.14) in order to account for the phase factor of the diverging wave.

The unknown functions $p^1(t)$ and $q^1(t)$ can be found from the boundary conditions on the surface of the paraboloid (formulae (6.26), (6.27) of Chapter 3).

$$2\frac{\partial P_{s-1}}{\partial v}+Q_{s-1} = 0, \quad 4Q^*_{s-1}+2s\frac{\partial Q_{s-1}}{\partial v}+sP_{s-1} = 0 \quad (\text{for } v = v_0), \tag{1.15}$$

where

$$P_{s-1} = P^0_{s-1}+P'_{s-1}, \quad Q_{s-1} = Q^0_{s-1}+Q'_{s-1} \tag{1.16}$$

are the potentials of the total field.

Inserting into the boundary conditions (1.15) the values of P_{s-1} and Q_{s-1}, expressed in terms of the potentials (1.11) and (1.14) we obtain the following equations for the determination of the functions $p^1(t)$ and $q^1(t)$

$$\left.\begin{aligned} & p^1(t)\left[\frac{1-s}{v}t\zeta_1(v, s-1, -t)+2t\zeta_1'(v, s-1, -t)\right]+ \\ & +q^1(t)s\zeta_1(v, s-1, -t) = \frac{1-s}{v}t\psi(v, s-1, -t)+ \\ & +2t\psi'(v, s-1, -t)+s\psi(v, s-1, -t) \quad (\text{for } v = v_0), \\ & p^1(t)t\zeta_1(v, s-1, -t)+q^1(t)\left[\frac{1-s}{v}s\zeta_1(v, s-1, -t)+\right. \\ & \left. +2s\zeta_1'(v, s-1, -t)-2t\zeta_1(v, s-1, -t)\right] \\ & = \frac{1-s}{v}s\psi(v, s-1, -t)+2s\psi'(v, s-1, -t)-t\psi(v, s-1, -t) \\ & \qquad (\text{for } v = v_0). \end{aligned}\right\} \tag{1.17}$$

In the solution of the system of equations (1.17), we will go over from functions with the argument $-t$ to functions with arguments $-t\pm i$; this can be done using the recurrence formulae (2.24) – (2.27) of Chapter 3. We obtain

$$\left.\begin{aligned} p^1(t) &= \frac{1}{D}\Big\{\psi(v, s, -t-i)\zeta_1(v, s, -t-i)+\frac{s^2+t^2}{4}\psi(v, s, -t+i)\times \\ &\times\zeta_1(v, s, -t+i)-\frac{i(s^2+t^2)}{2t}\left[\psi(v, s, -t+i)\zeta_1(v, s, -t-i)-\right. \\ &\left.-\psi(v, s, -t-i)\zeta_1(v, s, -t+i)\right]\Big\}, \\ q^1(t) &= \frac{1}{D}\Big\{\psi(v, s, -t-i)\zeta_1(v, s, -t-i)+ \\ &+\frac{s^2+t^2}{4}\psi(v, s, -t+i)\zeta_1(v, s, -t+i)\Big\}, \end{aligned}\right\} \tag{1.18}$$

where

$$D = \zeta_1^2(v, s, -t-i)+\frac{s^2+t^2}{4}\zeta_1^2(v, s, -t+i). \tag{1.19}$$

Having obtained from the formulae (1.11), (1.14) and (1.16), the potentials P_{s-1} and Q_{s-1} of the total field we find the functions C_{s-1}, D_{s-1}, F_s and G_s and then the field itself by means of the formulae (7.01) – (7.06) of Chapter 3.

We do not write out all the field components in explicit form, but confine ourselves to the values of the tangential components of the magnetic field on the paraboloid surface, which permit us to calculate the surface current density.

Using formulae (6.08) and (7.06) of Chapter 3, we obtain the following expression

$$\begin{aligned} 2iuH_u+H_\varphi &= \frac{1}{2}H_\varphi^{(0)}+\frac{1}{2}\sum_{s=1}^{\infty} i^{s-1}\sqrt{(uv)^s}\left\{2\frac{\partial D_{s-1}}{\partial v}+C_{s-1}\right\}e^{is\varphi}+ \\ &+\frac{1}{2}\sum_{s=1}^{\infty} i^{s-1}\sqrt{(uv)^s}\left\{C_{s-1}+2sG_s-2\frac{\partial D_{s-1}}{\partial v}\right\}e^{-is\varphi}. \end{aligned} \tag{1.20}$$

For the values of the Fourier coefficients on the surface of the paraboloid

(for $v=v_0$) we obtain

$$\left.\begin{aligned}
2\frac{\partial D_{s-1}}{\partial v}+C_{s-1} &= -\frac{2E_0}{\pi k\sin\delta}\sqrt{(uv)^{-s+1}}\int_{-\infty}^{+\infty}\frac{\left(\tan\frac{\delta}{2}\right)^{it}}{s^2+t^2}\times\\
&\quad\times r(t)t(s+it)\psi(u,s-1,t)\zeta_1(v,s+1,-t)\,\mathrm{d}t,\\
C_{s-1}+2sG_s-2\frac{\partial D_{s-1}}{\partial v} &= \frac{2E_0}{\pi k\sin\delta}\sqrt{(uv)^{-s+1}}\int_{-\infty}^{+\infty}\frac{\left(\tan\frac{\delta}{2}\right)^{it}}{s^2+t^2}\times\\
&\quad\times r(t)t(s-it)\psi(u,s+1,t)\zeta_1(v,s-1,-t)\,\mathrm{d}t,
\end{aligned}\right\} \tag{1.21}$$

where

$$r(t) = p^1(t)-q^1(t) = i\frac{s^2+t^2}{D}V \tag{1.22}$$

and

$$V = \psi(v,s,-t+i)\zeta_1(v,s,-t-i)-\zeta_1(v,s,-t+i)\psi(v,s,-t-i). \tag{1.23}$$

If we use the relations

$$\left.\begin{aligned}
\zeta_1(v,s,-t-i) &= v\zeta_1'(v,s,-t+i)+\frac{i(v-t)}{2}\zeta_1(v,s,-t+i),\\
\psi(v,s,-t-i) &= v\psi'(v,s,-t+i)+\frac{i(v-t)}{2}\psi(v,s,-t+i),
\end{aligned}\right\}, \tag{1.24}$$

$$\psi(v,s,-t+i) = \zeta_1(v,s,-t+i)+\frac{\Gamma\left(\frac{s-it}{2}\right)}{\Gamma\left(\frac{s+2+it}{2}\right)}\zeta_2(v,s,-t+i), \tag{1.25}$$

which result from formulae (2.17) – (2.23) of Chapter 3, then the function V in (1.23) takes the form

$$\begin{aligned}
V = -v\frac{\Gamma\left(\frac{s-it}{2}\right)}{\Gamma\left(\frac{s+2+it}{2}\right)}&[\zeta_1(v,s,-t+i)\zeta_2'(v,s,-t+i)-\\
&-\zeta_1'(v,s,-t+i)\zeta_2(v,s,-t+i)].
\end{aligned} \tag{1.26}$$

The quantity in the square brackets of expression (1.26) is a Wronskian,

equal to $-(1/v)\,e^{\pi t/2}$ (see (10.14) of Chapter 3). Hence

$$V = \frac{\Gamma\left(\frac{s-it}{2}\right)}{\Gamma\left(\frac{s+2+it}{2}\right)} e^{\frac{\pi t}{2}}. \tag{1.27}$$

Taking into account the values (1.22) and (1.27) and inverting at the same time the order of summation and integration in (1.20), we obtain for the tangential (covariant) components of the total magnetic field on the surface of the paraboloid the following final expression

$$\begin{aligned} 2iuH_u + H_\varphi \\ = \frac{1}{2} H_\varphi^{(0)} + \frac{E_0\sqrt{(uv)}}{2\pi k \sin\delta} \int_{-\infty}^{+\infty} \left(\tan\frac{\delta}{2}\right)^{it} \Bigg\{ \sum_{s=1}^{\infty} (-1)^s e^{-i\frac{\pi}{2}s} e^{is\varphi} (s+it) \times \\ \times \frac{V}{D} \psi(u, s-1, t)\zeta_1(v, s+1, -t) - \sum_{s=1}^{\infty} (-1)^s e^{-i\frac{\pi}{2}s} e^{-is\varphi} (s-it) \times \\ \times \frac{V}{D} \psi(u, s+1, t)\zeta_1(v, s-1, -t) \Bigg\} dt. \end{aligned} \tag{1.28}$$

The quantity $-2iuH_u + H_\varphi$ is obtained from (1.28) by replacing the argument φ by $-\varphi$; from the two formulae for $H_\varphi \pm 2iuH_u$ we obtain easily the quantities H_u and H_φ separately.

2. *Summation of the Series. Geometrical Optics*

Let us rewrite expression (1.21) in the form

$$2iuH_u + H_\varphi = \frac{E_0\sqrt{(uv)}}{2\pi k \sin\delta} \int_{-\infty}^{+\infty} e^{it \log \tan \frac{\delta}{2}} S\, dt, \tag{2.01}$$

where

$$S = f(0) + \sum_{s=1}^{\infty} (-1)^s f_1(s)\, e^{is\varphi} + \sum_{s=1}^{\infty} (-1)^s f_2(s)\, e^{-is\varphi}, \tag{2.02}$$

and

$$f(0) = f_1(0) = f_2(0). \tag{2.03}$$

The functions $f_1(s)$ and $f_2(s)$ have the form

$$\left.\begin{aligned} f_1(s) &= N(s)\, e^{\frac{\pi t}{2}} \frac{2}{\Gamma\left(\frac{-s-it}{2}\right)} \xi(u, s-1, t)\zeta_1(v, s+1, -t), \\ f_2(s) &= -N(s)\, e^{\frac{\pi t}{2}} \frac{s-it}{\Gamma\left(\frac{-s-it}{2}\right)} \xi(u, s+1, t)\zeta_1(v, s-1, -t), \end{aligned}\right\} \tag{2.04}$$

where the function ξ is defined by formula (2.07) of Chapter 3, and

$$N(s) = e^{-i\frac{\pi}{2}s}\frac{\Gamma\left(\frac{s-it}{2}\right)\Gamma\left(\frac{-s-it}{2}\right)}{D} \tag{2.05}$$

$N(s)$ is an even function of s. We consider the sum S for a fixed value of t, which we shall suppose to be positive. The nature of the discussion does not change, in principle, for $t<0$.

The functions $f_1(s)$ and $f_2(s)$ are holomorphic in the fourth quadrant. In the first and third quadrants these functions have poles which are zeroes of the denominator D. Moreover, there are poles in the second quadrant (Fig. 2), because of the factor $\Gamma(s-it)/2$ in $N(s)$.

The sum S can be represented in the form

$$S = f(0)+\frac{1}{2i}\int_C \frac{f_1(s)\,e^{is\varphi}+f_2(s)\,e^{-is\varphi}}{\sin \pi s}\,ds, \tag{2.06}$$

since the integral on the right-hand side reduces to residues at the points $s=1, 2, 3, \ldots$. The contour C can be replaced by the contours C_1 and C_2 (Fig. 2) and we can write

$$S = S_1+S_2 \tag{2.07}$$

where

$$\left.\begin{aligned} S_1 &= f(0)+\frac{1}{2i}\int_{C_1} \frac{f_1(s)\,e^{is\varphi}+f_2(s)\,e^{-is\varphi}}{\sin s\pi}\,ds, \\ S_2 &= \frac{1}{2i}\int_{C_2} \frac{f_1(s)\,e^{is\varphi}+f_2(s)\,e^{-is\varphi}}{\sin s\pi}\,ds. \end{aligned}\right\} \tag{2.08}$$

Let us transform the integral S_1. To do this, we shall need the value of the difference $f_1(-s)-f_2(s)$. Using formulae (2.16), (2.18), (2.29) of Chapter 3, we obtain

$$f_1(-s) -f_2(s) = i \sin s\pi\, g(s), \tag{2.09}$$

where

$$g(s) = \frac{i}{\pi}\,\frac{e^{\pi t}(s-it)}{D}\,\Gamma\left(\frac{s-it}{2}\right)\Gamma\left(\frac{-s-it}{2}\right)\zeta_1(u,\, s+1,\, t)\zeta_1(v,\, s-1,\, -t). \tag{2.10}$$

Inserting into the first equation (2.08) the value of $f_2(s)$ from (2.09), we find

$$S_1 = f(0)+\frac{1}{2i}\int_{C_1} \frac{f_1(s)\,e^{is\varphi}}{\sin s\pi}\,ds+\frac{1}{2i}\int_{C_1} \frac{f_1(-s)\,e^{-is\varphi}}{\sin s\pi}\,ds-\frac{1}{2}\int_{C_1} g(s)\,e^{-is\varphi}\,ds. \tag{2.11}$$

If in the second integral of (2.11) the variable s is replaced by $-s$, then it is easy to see that the sum of the first two integrals reduces to a residue at the point $s=0$ which is equal to $-f(0)$. Furthermore, since the function $g(s)$ has no poles on the real axis, we can draw the contour

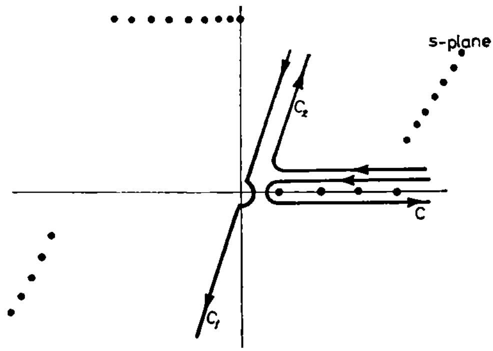

FIG. 2.

of integration of the third integral through the origin. If at the same time we change the direction of the integration along this contour, we obtain

$$S_1 = \frac{1}{2}\int_{C_0} g(s)\, e^{-is\varphi}\, ds, \tag{2.12}$$

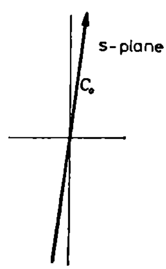

FIG. 3.

where the contour C_0 is a straight line making a small angle with the imaginary axis, as shown in Fig. 3. Formula (2.12) is valid under the condition that

$$-\frac{\pi}{2} < \varphi < \frac{\pi}{2}$$

It will be shown below, that to a very high approximation the main contribution to (2.01) is given by the integral of the function S_1 while the integral $\int e^{it \log \tan \frac{\delta}{2}} S_2 \, dt$ can be neglected. Consequently, in the subsequent calculations we will retain only S_1. The physical meaning of the contribution given by the function S_2 is apparently the same as in the case of a sphere (waves creeping around).

We now obtain the quantity (2.01) in the form

$$2iuH_u + H_\varphi = \frac{E_0 \sqrt{(uv)}}{4\pi k \sin \delta} \int\int g(s) \, e^{-is\varphi + it \log \tan \frac{\delta}{2}} \, ds \, dt \tag{2.13}$$

the limits of integration over t and over s are the same as in (2.01) and in (2.12) respectively.

The integral in (2.13) can be calculated approximately under the assumption that the quantity $v = v_0 = ka$ is very large.

First of all, we introduce instead of the functions ζ_1 new functions Z defined by means of the formulae

$$\left.\begin{aligned} \zeta_1(u, s, t) &= \frac{\sqrt{\pi}}{2} \sqrt{\left[\frac{\Gamma\left(\frac{s+1+it}{2}\right)}{\Gamma\left(\frac{s+1-it}{2}\right)}\right]} \, e^{-\frac{\pi t}{4}} Z(u, s, t), \\ \zeta_1(v, s, -t) &= \frac{\sqrt{\pi}}{2} \sqrt{\left[\frac{\Gamma\left(\frac{s+1-it}{2}\right)}{\Gamma\left(\frac{s+1+it}{2}\right)}\right]} \, e^{\frac{\pi t}{4}} Z(v, s, -t), \end{aligned}\right\} \tag{2.14}$$

Then the expression for $g(s)$ is somewhat simplified

$$g(s) = \frac{8 \, e^{-i(s+1)\frac{\pi}{2}}}{e^{-\pi t} - e^{-i\pi s}} \, \frac{Z(u, s+1, t) Z(v, s-1, -t)}{\sqrt{(s^2+t^2)} \left[Z^2(v, s, -t-i) - Z^2(v, s, -t+i)\right]}. \tag{2.15}$$

For large values of the parameter ka we replace the functions Z by their asymptotic expressions, obtained by the "semiclassical" method:

$$\left.\begin{aligned} Z(u, s+1, t) &= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt[4]{(u^2+2ut-s^2)}} \, e^{\frac{i}{2}\left(\chi_1 + \frac{\partial \chi_1}{\partial s} - \frac{\pi}{2}\right)}, \\ Z(v, s-1, -t) &= \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt[4]{(v^2-2vt-s^2)}} \, e^{\frac{i}{2}\left(\chi_1^* - \frac{\partial \chi_1^*}{\partial s} - \frac{\pi}{2}\right)}, \end{aligned}\right\} \tag{2.16}$$

where

$$\left.\begin{aligned}\chi_1 &= \int \sqrt{(u^2+2ut-s^2)}\,\frac{\mathrm{d}u}{u} = \sqrt{(u^2+2ut-s^2)} - s\,\arccos\frac{\dfrac{s^2}{u}-t}{\sqrt{(s^2+t^2)}} + \\ &\qquad + i\,\operatorname{arch}\frac{u+t}{\sqrt{(s^2+t^2)}},\\ \chi_1^* &= \int \sqrt{(v^2-2vt-s^2)}\,\frac{\mathrm{d}v}{v} = \sqrt{(v^2-2vt-s^2)} - s\,\arccos\frac{\dfrac{s^2}{v}+t}{\sqrt{(s^2+t^2)}} - \\ &\qquad - t\,\operatorname{arch}\frac{v-t}{\sqrt{(s^2+t^2)}}.\end{aligned}\right\} \tag{2.17}$$

(with $\operatorname{arch} x = \log(x+\sqrt{\{x^2-1\}})$).

The constant coefficients in the expressions (2.16), are determined by a comparison of these expressions with the formulae (2.11) and (2.13) of Chapter 3, which give asymptotic representations for the functions ζ_1 for large u, v. It can be also shown that

$$-\frac{8i}{\pi}\,\frac{1}{\sqrt{(s^2+t^2)}\,[Z^2(v,s,-t-i)-Z^2(v,s,-t+i)]} = \mathrm{e}^{-i\chi_1^*}. \tag{2.18}$$

Supposing that the relevant part of the integration over the variable t corresponds to large values of t, so that the quantity $e^{-\pi t}$ in the denominator of (2.15) can be neglected, we obtain the following expression for the function $g(s)$:

$$g(s) = 4\,\mathrm{e}^{i\frac{s\pi}{2}}\,\frac{\exp\left\{\dfrac{i}{2}\chi_1 - \dfrac{i}{2}\chi_1^* + \dfrac{i}{2}(\beta-\alpha+\pi)\right\}}{\sqrt[4]{(u^2+2ut-s^2)}\,\sqrt[4]{(v^2-2vt-s^2)}}, \tag{2.19}$$

where the following notations have been introduced:

$$\left.\begin{aligned}&\frac{\partial\chi_1}{\partial s} = -\arccos\frac{\dfrac{s^2}{u}-t}{\sqrt{(s^2+t^2)}} = -\alpha, \qquad \frac{\partial\chi_1^*}{\partial s} = -\arccos\frac{\dfrac{s^2}{v}+t}{\sqrt{(s^2+t^2)}} = -\beta,\\ &\frac{\partial\chi_1}{\partial t} = \operatorname{arch}\frac{u+t}{\sqrt{(s^2+t^2)}} = a, \qquad \frac{\partial\chi_1^*}{\partial t} = -\operatorname{arch}\frac{v-t}{\sqrt{(s^2+t^2)}} = -b.\end{aligned}\right\} \tag{2.20}$$

Denoting the phase of the integrand in (2.13) by

$$\omega = s\frac{\pi}{2} + \frac{1}{2}\chi_1 - \frac{1}{2}\chi_1^* - s\varphi + t\log\tan\frac{\delta}{2} \tag{2.21}$$

we find the stationary points from the conditions

$$\left.\begin{aligned}\frac{\partial\omega}{\partial s} &= \frac{\pi}{2}-\varphi-\frac{1}{2}\alpha+\frac{1}{2}\beta = 0,\\ \frac{\partial\omega}{\partial t} &= \frac{1}{2}a+\frac{1}{2}b+\log\tan\frac{\delta}{2} = 0.\end{aligned}\right\} \tag{2.22}$$

Solving (2.22) with respect to s and t, we obtain the roots

$$\left.\begin{aligned}s^{(0)} &= \sqrt{(uv)}\sin\delta\sin\varphi,\\ t^{(0)} &= \frac{1}{2}(v-u)\sin^2\delta+\sqrt{(uv)}\sin\delta\cos\delta\cos\varphi.\end{aligned}\right\} \tag{2.23}$$

The "semi-classical" method is only applicable if the relation

$$\sqrt{(v^2-2vt^{(0)}-s^{(0)^2})} = v\cos\delta-\sqrt{(uv)}\sin\delta\cos\varphi \gg 1 \tag{2.24}$$

is satisfied.

Since $v\cos\delta-\sqrt{(uv)}\sin\delta\cos\varphi = 0$ is the equation of the boundary of the shadow, the formulae which we obtain by the stationary phase method will be valid in the illuminated region far enough away from the shadow boundary.

From (2.17) and (2.21), the following expression for the phase can be obtained:

$$\omega = -\left[\frac{1}{2}\sqrt{(v^2-2vt-s^2)}-\frac{1}{2}\sqrt{(u^2+2ut-s^2)}\right]+s\left(\frac{\pi}{2}-\varphi-\frac{\alpha}{2}+\frac{\beta}{2}\right)+$$
$$+t\left(\log\tan\frac{\delta}{2}+\frac{1}{2}a+\frac{1}{2}b\right) = -\left[\frac{1}{2}\sqrt{(v^2-2vt-s^2)}-\frac{1}{2}\sqrt{(u^2+2ut-s^2)}\right]+$$
$$+s\frac{\partial\omega}{\partial s}+t\frac{\partial\omega}{\partial t}. \tag{2.25}$$

Now we make a contact transformation (Legendre transformation), with the transformation function ω, from the variables s, t to the variables p, q defined by

$$\frac{\partial\omega}{\partial s} = p, \qquad \frac{\partial\omega}{\partial t} = q. \tag{2.26}$$

Denoting

$$\frac{1}{2}\sqrt{(v^2-2vt-s^2)}-\frac{1}{2}\sqrt{(u^2+2ut-s^2)} = \Omega(p,q), \tag{2.27}$$

we have

$$\omega = sp+tq-\Omega(p, q). \tag{2.28}$$

Owing to the reciprocity of the Legendre transformation, the following relations hold

$$s = \frac{\partial \Omega}{\partial p}, \qquad t = \frac{\partial \Omega}{\partial q}, \tag{2.29}$$

which can also be derived directly.

We introduce formally the quantities φ_1 and δ_1 by the formulae

$$\varphi_1 = \varphi+p, \quad \log \tan \frac{\delta_1}{2} = \log \operatorname{tg} \frac{\delta}{2} - q, \tag{2.30}$$

These quantities satisfy the relations

$$\left.\begin{aligned} &\frac{\partial \varphi_1}{\partial p} = 1, \qquad \frac{\partial \delta_1}{\partial q} = -\sin \delta_1, \\ &\cos \delta_1 = \frac{\sinh q + \cosh q \cos \delta}{\cosh q + \sinh q \cos \delta} \qquad \sin \delta_1 = \frac{\sin \delta}{\cosh q + \sinh q \cos \delta}, \end{aligned}\right\} \tag{2.31}$$

with whose help s and t can be written as functions of p and q, namely

$$\left.\begin{aligned} s &= \sqrt{(uv)} \sin \delta_1 \sin \varphi_1, \\ t &= \frac{1}{2}(v-u)\sin^2 \delta_1 + \sqrt{(uv)} \sin \delta_1 \cos \delta_1 \cos \varphi_1, \\ \Omega &= \frac{1}{2}(v-u)\cos \delta_1 - \sqrt{(uv)} \sin \delta_1 \cos \varphi_1. \end{aligned}\right\} \tag{2.32}$$

It is evident that for $p=0$, $q=0$ when $\varphi_1=\varphi$ and $\delta_1=\delta$ the formulae (2.32) transform into (2.23), as they ought to do. Calculating from (2.32) the values of the partial derivatives of the old coordinates (s, t) with respect to the new ones (p, q), we find the Jacobian of the contact transformation

$$\frac{D(s, t)}{D(p, q)} = \frac{\partial s}{\partial p}\frac{\partial t}{\partial q} - \frac{\partial s}{\partial q}\frac{\partial t}{\partial p} = -\sin^2 \delta_1 \big(u \cos \delta_1 + \sqrt{(uv)} \sin \delta_1 \cos \varphi_1\big) \times$$

$$\times \big(v \cos \delta_1 - \sqrt{(uv)} \sin \delta_1 \cos \varphi_1\big). \tag{2.33}$$

Transforming the integral in (2.13) to the new variables p and q, and

keeping in mind the equality

$$\frac{1}{2}(\beta-\alpha+\pi) = \varphi+p\mp\varphi_1, \tag{2.34}$$

we obtain

$$2iuH_u+H_\varphi = \frac{E_0\sqrt{(uv)}}{\pi k\sin\delta}\int\int e^{i\omega_1(p,\,q)+i\varphi_1(p)}\sin\delta_1\sqrt{\left\{-\frac{D(s,t)}{D(p,q)}\right\}}\,dp\,dq. \tag{2.35}$$

When transforming the integrand in (2.13), we used the relation

$$\frac{1}{\sqrt[4]{(v^2-2vt-s^2)}\sqrt[4]{(u^2+2ut-s^2)}} = \frac{\sin\delta_1}{\sqrt{\left[-\dfrac{D(s,t)}{D(p,q)}\right]}}. \tag{2.36}$$

The quantity ω_1 is the phase (2.28) considered as a function of p and q.

We evaluate the integral (2.35) by the method of stationary phase. The stationary conditions will be

$$\frac{\partial\omega_1}{\partial p} = 0, \qquad \frac{\partial\omega_1}{\partial q} = 0. \tag{2.37}$$

For the stationary phase point the following system of equations is obtained:

$$p\frac{\partial s}{\partial p}+q\frac{\partial t}{\partial p} = 0, \qquad p\frac{\partial s}{\partial q}+q\frac{\partial t}{\partial q} = 0. \tag{2.38}$$

This system has the unique solution $p=0$, $q=0$ since the determinant of the system (2.33) is not zero.

Let us expand $\omega_1(p, q)$ in the vicinity of the point $p=0$, $q=0$ in a series, up to terms of the second order

$$\omega_1(p,q) = -\Omega(0,0)+\frac{1}{2}\left\{\frac{\partial^2\omega_1}{\partial p^2}p^2+2\frac{\partial^2\omega_1}{\partial p\,\partial q}pq+\frac{\partial^2\omega_1}{\partial q^2}q^2\right\}. \tag{2.39}$$

The second derivatives of ω_1 with respect to p and q are taken at the point $p=q=0$, where they are equal to

$$\frac{\partial^2\omega_1}{\partial p^2} = \frac{\partial s}{\partial p}, \qquad \frac{\partial^2\omega_1}{\partial p\,\partial q} = \frac{\partial s}{\partial q} = \frac{\partial t}{\partial p}, \qquad \frac{\partial^2\omega_1}{\partial q^2} = \frac{\partial t}{\partial q}. \tag{2.40}$$

Substituting the expansion (2.39) into (2.35) and calculating the integral

approximately by the usual method, we find

$$2iuH_u+H_\varphi = \frac{E_0\sqrt{(uv)}}{\pi k}\,\mathrm{e}^{-i\Omega(0,\,0)}\,\sigma, \tag{2.41}$$

where $-\Omega(0,\ 0) = \frac{1}{2}(u-v)\cos\delta+\sqrt{(uv)}\sin\delta\cos\varphi$ is the phase of the incident plane wave and σ is the factor

$$\sigma = \sqrt{\left(-\frac{D(s,\,t)}{D(p,\,q)}\right)}\iint \mathrm{e}^{\frac{i}{2}\left\{\frac{\partial s}{\partial p}p^2+2\frac{\partial s}{\partial q}pq+\frac{\partial t}{\partial q}q^2\right\}}\,\mathrm{d}p\,\mathrm{d}q = 2\pi. \tag{2.42}$$

Thus

$$2iuH_u+H_\varphi = \frac{2E_0\sqrt{(uv)}}{k}\,\mathrm{e}^{-i\Omega(0,\,0)}\,\mathrm{e}^{i\varphi}. \tag{2.43}$$

In accordance with geometrical optics, we obtain for the value of the total field on the surface of the paraboloid twice the value of the field of the incident wave.

3. *Asymptotic Formulae for the Field in the Penumbra Region*

Now, let us turn to the determination of the field in the penumbra region. Under the condition

$$v^2-2vt-s^2 \lessdot [2(v^2+s^2)]^{2/3}, \tag{3.01}$$

that is, when the quantity $v^2-2vt-s^2$ is small in comparison with each of the three terms composing this quantity, the asymptotic formulae (2.16) and (2.18) for the functions of the coordinate v are not applicable. If condition (3.01) is satisfied the following equality holds approximately:

$$\chi_1^* = \frac{2}{3(v^2+s^2)}(v^2-2vt-s^2)^{3/2}, \tag{3.02}$$

and the parabolic functions can be expressed asymptotically in terms of the Airy functions. We have

$$\left.\begin{aligned} Z(v,\,s-1,\,-t) &= -\frac{2i}{\sqrt{\pi}}[2(v^2+s^2)]^{-1/6}\,w(\tau-s_1),\\ \sqrt{(s^2+t^2)}\,[Z^2(v,\,s,\,-t-i)-Z^2(v,\,s,\,-t+i)] &= -\frac{2i}{\pi}\,w(\tau)w'(\tau), \end{aligned}\right\} \tag{3.03}$$

where

$$\left.\begin{aligned} \tau &= -\left(\frac{3}{4}\chi_1^*\right)^{2/3} = -\frac{1}{[2(v^2+s^2)]^{2/3}}(v^2-2vt-s^2),\\ s_1 &= \frac{\partial\tau}{\partial s} = \frac{2s}{[2(v^2+s^2)]^{2/3}}\,. \end{aligned}\right\} \tag{3.04}$$

From (3.03) we obtain

$$\mathrm{d}t = \frac{1}{v_1}\mathrm{d}\tau, \tag{3.05}$$

where

$$v_1 = \frac{\partial \tau}{\partial t} = \frac{2v}{[2(v^2+s^2)]^{2/3}}. \tag{3.06}$$

The asymptotic expression for the function $Z(u, s+1, t)$ is given by (2.16) as before. However, owing to condition (3.01), we can use the approximate equality

$$u^2+2ut-s^2 \simeq (u+v)^2 \cos^2 \delta_1. \tag{3.07}$$

Consequently, the first formula (2.16) becomes

$$Z(u, s+1, t) = \frac{2}{\sqrt{\pi}\sqrt{(u+v)}\sqrt{(\cos \delta_1)}} e^{i\frac{\chi_1}{2} - \frac{i}{2}\alpha - i\frac{\pi}{4}}, \tag{3.08}$$

The equalities (3.03) and (3.08) yield the follo wing expression for the function $g(s)$

$$g(s) = \frac{4[2(v^2+s^2)]^{-1/6}}{\sqrt{(u+v)}\sqrt{(\cos \delta_1)}} \frac{w(\tau-s_1)}{w(\tau)w'(\tau)} e^{\frac{i}{2}\left(\chi_1-\alpha+s\pi+\frac{\pi}{2}\right)}. \tag{3.09}$$

Comparing (3.09) with (2.19), we see that the new phase in the integrand of (2.13)

$$\tilde{\omega} = \frac{1}{2}\chi_1+s\left(\frac{\pi}{2}-\varphi\right)+t \log \tan \frac{\delta}{2} \tag{3.10}$$

is connected with the old phase (2.21) by the relation

$$\tilde{\omega} = \omega+\frac{1}{2}\chi_1^*. \tag{3.11}$$

We expand $\tilde{\omega}$ as a series at $\chi_1^* = 0$ limiting ourselves to a constant and a linear term of the expansion

$$\tilde{\omega} = \tilde{\omega}\Big|_{\chi_1^*=0} + \frac{\partial \tilde{\omega}}{\partial t}\Big|_{\chi_1^*=0} \frac{\partial t}{\partial \tau}\cdot\tau. \tag{3.12}$$

It is evident that when $\chi_1^*=0$ the derivative $\partial\chi_1^*/\partial t$ also vanishes (see (3.02)). Consequently, for $\chi_1^*=0$ the following equalities are valid:

$$\tilde{\omega} = \omega, \qquad \frac{\partial \tilde{\omega}}{\partial t} = \frac{\partial \omega}{\partial t} \tag{3.13}$$

and therefore

$$\tilde{\omega} = \omega\Big|_{\chi_1^*=0} + \frac{\partial \omega}{\partial t}\Big|_{\chi_1^*=0}\frac{1}{v_1}\tau. \tag{3.14}$$

We introduce by means of the contact transformation the variables p and q and the function Ω defined by

$$\omega = sp+tq-\Omega(p, q) \tag{3.15}$$

and we expand the function ω in a series near the point $p=p_0$, $q=q_0$ where $\chi_1^*=0$ and

$$\mathrm{d}\omega \equiv p\,\mathrm{d}s+q\,\mathrm{d}t = 0 \tag{3.16}$$

From the relation (3.16) and the equality

$$\mathrm{d}\chi_1^* = -\beta\,\mathrm{d}s-b\,\mathrm{d}t = 0 \tag{3.17}$$

we find using (3.02) an equation connecting p_0 and q_0, namely

$$p_0-q_0\frac{s}{v} = 0 \tag{3.18}$$

The condition $\chi_1^*=0$ means that

$$v\cos\delta_1-\sqrt{(uv)}\sin\delta_1\cos\varphi_1 = 0. \tag{3.19}$$

With help of (2.31), we obtain the following approximate relation valid for p and q small in comparison to unity

$$v\cos\delta-\sqrt{(uv)}\sin\delta\cos\varphi+sp+vq = 0. \tag{3.20}$$

Formula (3.20) holds, in particular, for $p=p_0$, $q=q_0$. Using it together with equation (3.18) we find

$$\left.\begin{aligned}(s^2+v^2)p_0 &= -s\big(v\cos\delta-\sqrt{(uv)}\sin\delta\cos\varphi\big),\\ (s^2+v^2)q_0 &= -v\big(v\cos\delta-\sqrt{(uv)}\sin\delta\cos\varphi\big).\end{aligned}\right\} \tag{3.21}$$

Now we expand the phase ω in powers of $p-p_0$ and $q-q_0$:

$$\begin{aligned}\omega = \omega_0&+\left[p\frac{\partial s}{\partial p}+q\frac{\partial t}{\partial p}\right]_0(p-p_0)+\left[p\frac{\partial s}{\partial q}+q\frac{\partial t}{\partial q}\right]_0(q-q_0)+\\ &+\frac{1}{2}\left[p\frac{\partial^2 s}{\partial p^2}+q\frac{\partial^2 t}{\partial p^2}+\frac{\partial s}{\partial p}\right]_0(p-p_0)^2+\left[p\frac{\partial^2 s}{\partial p\,\partial q}+q\frac{\partial^2 t}{\partial p\,\partial q}+\frac{\partial t}{\partial p}\right]_0\times\\ &\times(p-p_0)(q-q_0)+\frac{1}{2}\left[p\frac{\partial^2 s}{\partial q^2}+q\frac{\partial^2 t}{\partial q^2}+\frac{\partial t}{\partial q}\right]_0(q-q_0)^2.\end{aligned} \tag{3.22}$$

In formula (3.21) the linear terms of the expansion vanish since we have

$$\left.\begin{aligned} p_0\left(\frac{\partial s}{\partial p}\right)_0+q_0\left(\frac{\partial t}{\partial p}\right)_0 &= (p_0v-q_0s)\cos\delta_1 = 0,\\ p_0\left(\frac{\partial s}{\partial q}\right)_0+q_0\left(\frac{\partial t}{\partial q}\right)_0 &= \frac{\left(\frac{\partial s}{\partial q}\right)_0}{\left(\frac{\partial s}{\partial p}\right)_0}\left[p_0\left(\frac{\partial s}{\partial p}\right)_0+q_0\left(\frac{\partial t}{\partial p}\right)_0\right] = 0. \end{aligned}\right\} \tag{3.23}$$

When the values of the partial derivatives at p_0, q_0 are inserted into (3.22), the second order terms therein reduce to an exact square and we have

$$\omega = \omega_0+\frac{1}{2v}(\sec\delta_1-q_0)(s-s_0)^2, \tag{3.24}$$

where s_0 is the value of s at $p=p_0$, $q=q_0$.

We now expand $\Omega(p, q)$ in a series of powers of $(p-p_0)$ and $(q-q_0)$ up to terms of third order, and put $p=0$, $q=0$ in the expansion obtained

$$\begin{aligned}\Omega(0, 0) = \Omega(p_0, q_0)-p_0s_0-q_0t_0+\frac{1}{2}\Bigg[\left(\frac{\partial s}{\partial p}\right)_0 p_0^2+2\left(\frac{\partial t}{\partial p}\right)_0 p_0q_0+\\ +\left(\frac{\partial t}{\partial q}\right)_0 q_0^2\Bigg]-\frac{1}{6}\Bigg[p_0^3\left(\frac{\partial^3\Omega}{\partial p^3}\right)_0+3p_0^2q_0\left(\frac{\partial^3\Omega}{\partial p^2\,\partial q}\right)_0+\\ +3p_0q_0^2\left(\frac{\partial^3\Omega}{\partial p\,\partial q^2}\right)_0+q_0^3\left(\frac{\partial^3\Omega}{\partial q^3}\right)_0\Bigg].\end{aligned} \tag{3.25}$$

Since $\partial s/\partial q=\partial t/\partial p$, then the quadratic form in (3.25) is proportional to the square of the quantity $p_0(\partial s/\partial p)_0+q_0(\partial s/\partial q)_0$ which vanishes. Consequently, besides the quantity

$$\Omega(p_0, q_0)-p_0s_0-q_0t_0 = -\omega_0, \tag{3.26}$$

there remains in (3.25) a component of third order in the variables p_0, q_0 which, after calculation of the third order derivatives involved, reduces to

$$\frac{1}{6}\left[p_0^3\left(\frac{\partial^3\Omega}{\partial p^3}\right)_0+\dots\right] = -\frac{(\sqrt{(uv)}\sin\delta\cos\varphi-v\cos\delta)^3}{6v(u+v)\sin^2\delta_1} = -\frac{\xi^3}{3}, \tag{3.27}$$

where

$$\xi = \frac{\sqrt{(uv)}\sin\delta\cos\varphi-v\cos\delta}{[2v(u+v)\sin^2\delta_1]^{1/3}}. \tag{3.28}$$

Substituting the value of ω_0 from (3.25) and (3.26) in the formula (3.24), and using (3.28) we find

$$\omega = -\Omega(0, 0)+\frac{\xi^3}{3}+\frac{1}{2v}(\sec\delta_1-q_0)(s-s_0)^2. \tag{3.29}$$

The quantity $\tilde{\omega}$ is connected with ω by means of formula (3.14) which contains the term

$$\left.\frac{\partial\omega}{\partial t}\right|_{x_1^*=0}\frac{1}{v_1}\tau = \frac{q_0}{v_1}\tau = \frac{\sqrt{(uv)}\sin\delta\cos\varphi-v\cos\delta}{[2(v^2+s^2)]^{1/3}}\tau. \tag{3.30}$$

Because of the approximate relation

$$v^2+s^2 \simeq v(v+u)\sin^2\delta_1 \tag{3.31}$$

this term is equal to

$$\frac{q_0}{v_1}\tau = \frac{\sqrt{(vu)}\sin\delta\cos\varphi-v\cos\delta}{[2v(u+v)\sin^2\delta_1]^{1/3}}\tau = \xi\tau. \tag{3.32}$$

Consequently, we will have $\tilde{\omega}=\omega+\xi\tau$ or

$$\tilde{\omega} = -\Omega(0, 0)+\frac{\xi^3}{3}+\frac{1}{2v}(\sec\delta_1-q_0)(s-s_0)^2+\xi\tau. \tag{3.33}$$

Thus the phase in the integral (2.13) is expressed in terms of s and τ. Evaluating the integral over s (it reduces to the Fresnel integral), we obtain

$$2iuH_u+H_\varphi = \frac{E_0\sqrt{(uv)}}{k}e^{-i\Omega(0,0)}\sqrt{\left(\frac{\sin\delta_1}{\sin\delta}\right)}e^{i(\varphi+p_0)}\frac{e^{i\frac{\xi^3}{3}}}{\sqrt{\pi}}\int_\Gamma\frac{w(\tau-s_1)}{w(\tau)w'(\tau)}e^{i\xi\tau}\,d\tau, \tag{3.34}$$

where Γ is a contour running along the line arc $\tau=2\pi/3$ from infinity to zero and then along the real axis to infinity.

If $\pi/2-\varphi$ is small (of the order of $v^{-1/3}$ or less), then the derivation of formula (3.34) must be modified, and another asymptotic expression must be taken for $Z(u, s+1, t)$. The resulting expression is, however, identical with the limiting form of the preceding one.

In the penumbra region, when $\xi\sim 1$ the quantities p_0, q_0 are of the order $v^{-1/3}$ as follows from the relations (3.21), (3.32). Consequently, we can neglect these quantities without introducing a large error. Neglecting at the same time the small quantity s_1 we can write formula (3.34) in the form

$$2iuH_u+H_\varphi = \frac{E_0\sqrt{(uv)}}{k}e^{i\varphi-i\Omega(0,0)}G(\xi), \tag{3.35}$$

where

$$G(\xi) = e^{i\frac{\xi^3}{3}} \frac{1}{\sqrt{\pi}} \int_\Gamma \frac{e^{i\xi\tau}}{w'(\tau)} d\tau. \tag{3.36}$$

These formulae were first obtained in Ref. 2 (formulae (3.17)–(3.19) of Chapter 2).

In the region of large positive values of ξ (deep shade) the quantities we neglected are not essential because of the exponential decay of the function $G(\xi)$. In the region of large negative values of ξ this neglect is justified by the fact that formula (3.36) is in agreement with the formula of geometrical optics.

It should be noted that if we calculate the integral $\int S_2 e^{it \log \tan \delta/2} dt$ using the same approximation as in the integral of (2.13), we obtain zero for its value because of the situation of the contour C_2. This justifies neglecting the quantity S_2.

4. *Exact Solution and Asymptotic Formulae for the Second Type of Polarization of the Incident Wave*

In Section 1 we obtained an exact solution for the first type of polarization of the plane wave. In order to obtain asymptotic formulae applicable for arbitrary polarization of the incident wave, we must first find the exact solution for the second type of polarization when the electric vector **E** of the external field is directed along the y axis (Fig. 1).

In this case, the Cartesian components of the plane wave field are

$$\left.\begin{aligned} E_x &= 0, & H_x &= -E_0 \cos\delta\, e^{i\Omega}, \\ E_y &= E_0 e^{i\Omega} & H_y &= 0, \\ E_z &= 0, & H_z &= E_0 \sin\delta\, e^{i\Omega}, \end{aligned}\right\} \tag{4.01}$$

and the covariant components of the tangential magnetic field are determined from the expressions

$$\left.\begin{aligned} 2iuH_u + H_\varphi &= -i\frac{E_0}{k}\sqrt{(uv)}\cos\delta\, e^{i\Omega+i\varphi} + i\frac{E_0}{k} u \sin\delta\, e^{i\Omega}, \\ -2iuH_u + H_\varphi &= i\frac{E_0}{k}\sqrt{(uv)}\cos\delta\, e^{i\Omega-i\varphi} - i\frac{E_0}{k} u \sin\delta\, e^{i\Omega}. \end{aligned}\right\} \tag{4.02}$$

Comparing the expressions (4.01) with the formulae (1.02), we see that the fields of the plane waves in the two cases under consideration are connected by of the relations

$$\mathbf{E}^{(2)} = \mathbf{H}^{(1)} \qquad \mathbf{H}^{(2)} = -\mathbf{E}^{(1)} \tag{4.03}$$

where the superscripts "1" and "2" refer, respectively, to the first and to the second types of polarization of the plane wave.

Taking the symmetry of the Maxwell equations in free space into account, we can express, in the case of the second type of polarization, the field quantities **E** and **H** of the total field in terms of the auxiliary functions C_{s-1}, D_{s-1}, F_s, G_s in the same way as the quantities **H** and $-$**E** were expressed in the case of the first type of polarization.

Consequently the expressions (7.05) and (7.06) of Chapter 3 are modified as follows:

$$\left.\begin{aligned}
2iuE_u^{(s)} &= \sqrt{(uv)^s}\left(2\frac{\partial D_{s-1}}{\partial v}-sG_s\right),\\
2ivE_v^{(s)} &= \sqrt{(uv)^s}\left(-2\frac{\partial D_{s-1}}{\partial u}-sG_s\right),\\
iE_\varphi^{(s)} &= \sqrt{(uv)^s}\,(C_{s-1}+sG_s),\\
2uH_u^{(s)} &= -\sqrt{(uv)^s}\left(2\frac{\partial C_{s-1}}{\partial v}-sF_s\right),\\
2vH_v^{(s)} &= -\sqrt{(uv)^s}\left(-2\frac{\partial C_{s-1}}{\partial u}-sF_s\right),\\
H_\varphi^{(s)} &= -\sqrt{(uv)^s}\,(D_{s-1}-sF_s).
\end{aligned}\right\} \tag{4.04}$$

It is easy to see that the values of the auxiliary functions are exactly the same as in Section 1, in particular

$$\left.\begin{aligned}
C_{s-1}^0 &= \frac{2E_0}{k}\sqrt{(uv)^{-s+1}}\,e^{\frac{i}{2}(u-v)\cos\delta}\,J_{s-1}\left(\sqrt{(uv)}\sin\delta\right),\\
D_{s-1}^0 &= 0,
\end{aligned}\right\} \tag{4.05}$$

therefore, the previous expressions (1.11) for the potentials P_{s-1}^0 and Q_{s-1}^0 also remain valid.

The boundary conditions will, of course, change and become

$$\left.\begin{aligned}
2\frac{\partial Q_{s-1}}{\partial v}-P_{s-1} &= 0 \quad (\text{for } v=v_0),\\
4P_{s-1}^*+2s\frac{\partial P_{s-1}}{\partial v}-sQ_{s-1} &= 0 \quad (\text{for } v=v_0).
\end{aligned}\right\} \tag{4.06}$$

Comparing these relations with the formulae (1.15), we see that the new boundary conditions can be obtained from the old ones by a formal substitution of Q for P and of $-P$ for Q

$$P \to Q, \quad Q \to -P. \tag{4.07}$$

The solution of the equations (4.06) yields the following values for the coefficients of the secondary waves $p^1(t)$ and $q^1(t)$ (see (1.23)).

$$\left.\begin{aligned}
p^1(t) &= \frac{1}{D}\Bigg[\zeta_1(v, s, -t-i)\psi(v, s, -t-i)+\frac{i(s^2+t^2)}{2t}V+\\
&\quad+\frac{s^2+t^2}{4}\psi(v, s, -t+i)\zeta_1(v, s, -t+i)\Bigg],\\
q^1(t) &= \frac{1}{D}\Bigg[\psi(v, s, -t-i)\zeta_1(v, s, -t-i)+\\
&\quad+\frac{s^2+t^2}{4}\psi(v, s, -t+i)\zeta_1(v, s, -t+i)\Bigg].
\end{aligned}\right\} \qquad (4.08)$$

These expressions differ from (1.18) only in the sign of the V-term. The remaining argument is essentially the same as that of Sections 1–3. Therefore the result can be written at once in form of a double integral:

$$2iuH_u+H_\varphi = -\frac{E_0\sqrt{(uv)}}{2\pi k\sin\delta}\int\int h(s, t)\,\mathrm{e}^{-is\varphi+it\log\tan\frac{\delta}{2}}\,\mathrm{d}s\,\mathrm{d}t. \qquad (4.09)$$

The quantity $-2iuH_u+H_\varphi$, as can be shown, is obtained if the sign of the integral in (4.09) is reversed and if φ is replaced by $-\varphi$. The function $h(s, t)$ has the following value

$$h(s, t) = \frac{i}{\pi}\mathrm{e}^{\pi t}(s-it)\,\frac{\Gamma\left(\frac{-s-it}{2}\right)\Gamma\left(\frac{s-it}{2}\right)}{D}\,\zeta_1(u, s+1, t)\times$$
$$\times\left[\zeta_1'(v, s-1, -t)+\frac{1-s}{2v}\zeta_1(v, s-1, -t)\right] \qquad (4.10)$$

It differs from $g(s, t)$ of (2.14) only in that instead of $\zeta_1(v, s-1, -t)$ the more complicated expression

$$\left[\zeta_1'(v, s-1, -t)+\frac{1-s}{2v}\zeta_1(v, s-1, -t)\right]$$

enters therein.

This circumstance makes it possible to calculate the integral in (4.09) along the same lines as the integral (2.15).

Reducing the expression $h(s, t)$ to Z functions, we find

$$h(s, t) = 8\,\mathrm{e}^{i\frac{s+1}{2}\pi}\,\frac{Z(u, s+1, t)\left[Z'(v, s-1, -t)+\frac{1-s}{2v}Z(v, s-1, -t)\right]}{\sqrt{(s^2+t^2)}\left[Z^2(v, s, -t-i)-Z^2(v, s, -t+i)\right]} \qquad (4.11)$$

The function Z' is expressed asymptotically in terms of Airy functions as follows:

$$Z'(v, s-1, -t) = \frac{i}{\sqrt{\pi \cdot v}} [2v(u+v)\sin^2\delta_1]^{\frac{1}{6}} w'(\tau - s_1). \tag{4.12}$$

Neglecting unity in comparison with s and using formula (3.31), we can write

$$\frac{1-s}{2v} \simeq -\frac{\sqrt{\{v(u\sin^2\delta_1 - v\cos^2\delta_1)\}}}{2v}. \tag{4.13}$$

By virtue of condition (3.19), the relation (4.13) can be written in the form

$$\frac{1-s}{2v} \simeq -\frac{\sqrt{(uv)}\sin\delta_1\sin\varphi_1}{2v}. \tag{4.14}$$

Substituting the expressions (4.12) and (4.14) and performing transformations similar to those which were made in Section 3, we find the value of the double integral in (4.09)

$$2iuH_u + H_\varphi = \frac{E_0\sqrt{(uv)}}{k} e^{-i\Omega(0,0)} e^{i\varphi} G(\xi) \frac{\sqrt{(uv)}\sin\delta\sin\varphi}{v} +$$

$$+\frac{E_0\sqrt{(uv)}}{k} e^{-i\Omega(0,0)} e^{i\varphi} F(\xi) \frac{[2v(u+v)\sin^2\delta]^{1/3}}{v}, \tag{4.15}$$

where

$$F(\xi) = e^{i\frac{\xi^3}{3}} \frac{1}{\sqrt{\pi}} \int_\Gamma \frac{e^{i\xi\tau}}{w(\tau)} d\tau, \tag{4.16}$$

and Γ is the same contour as in formula (3.36).

In the illuminated region far enough away from the shadow boundary, the functions $F(\xi)$ and $G(\xi)$ can be replaced by their asymptotic expressions valid for large negative values of ξ:

$$F(\xi) = 2i\xi, \quad G(\xi) = 2. \tag{4.17}$$

Then the expressions $2iuH_u + H_\varphi$ and $-2iuH_u + H_\varphi$ go over into the well-known formulae of geometrical optics, i.e., they become equal to twice the value of expressions (4.02).

Formulae (3.35) and (4.15) permit the distribution of currents on a paraboloid to be found for the case of an incident plane wave of arbitrary polarization.

CHAPTER 5

THE FIELD OF A PLANE WAVE NEAR THE SURFACE OF A CONDUCTING BODY†

Abstract — Approximate formulae are derived for the field induced by an incident plane wave on and near the surface of a convex body of finite conductivity. Since these formulae give also the current distribution in the skin-layer on the surface, they can be used for the calculation (by means of definite integrals) of the field at arbitrary distances from the body, yielding thus an approximate solution of the general diffraction problem.

INTRODUCTION

In Chapter 2 the following result has been obtained. The values of the tangential components of the total magnetic field on the surface of a perfect conductor are equal to the surface values of the tangential components of the field of the incident wave multiplied by a certain universal function $G(\xi)$, depending on the argument $\xi = l/d$, where l is the distance from the geometrical boundary of the shadow, measured in the plane of incidence and d the width of the penumbra region. The quantity d is equal to $d = \sqrt[3]{\left(\frac{\lambda}{\pi} R_0^2\right)}$ where λ is the wave length and R_0 the radius of curvature of the surface in the plane of incidence. The surface current density being proportional and directed at right angles to the magnetic field, this result immediately gives the current distribution on the surface. This enables the amplitude of the scattered wave to be calculated.

In this paper we intend to generalize this result in several respects. Firstly, we shall find the field distribution not only on the surface of the body, but also in its neighbourhood (at distances that are small compared with the curvature radii of the surface). Secondly, we shall not consider the body to be a perfect conductor but shall regard it instead as a good conductor only, in the sense that on its surface the Leontovich conditions for

† Fock, 1946. The text is revised to include corrections indicated in Ref. 6.

the tangential field components are valid. We shall also show that a similar result holds for the normal component of the magnetic field. Explicit expressions, including correction terms, will be given for all six components of the electro-magnetic field.

The method we shall use will also differ from that used in the previous paper. In the previous paper we obtained our result by making use of the local character of the field in the penumbra region. We started from the exact solution of the problem for a particular case and then performed the approximate summation of the series. By the principle of the local field the result could be applied to the general case also. Now we shall find the solution directly for the general case of an arbitrary surface, using the method of parabolic equation proposed by Leontovich and developed in Chapter 11 for the case of a point source (dipole), situated on a plane or on a spherical surface.

1. *The Geometrical Aspect of the Problem*

Consider a convex body and a plane wave incident in the direction of the x-axis. If the equation of the surface of the body is

$$f(x, y, z) = 0 \tag{1.01}$$

then the equation of the curve, representing the boundary of the geometrical shadow on the surface, will be obtained from the equation of the surface and the relation

$$\frac{\partial f}{\partial x} = 0 \tag{1.02}$$

Let us take on the surface a point lying on the boundary of the geometrical shadow and consider it to be the origin of our coordinate system. The $z-$axis we direct along the normal to the surface (towards the air). Since on the shadow boundary the normal is perpendicular to the direction of the wave, the z-axis so chosen will be perpendicular to the x-axis. The direction of the y-axis we choose in such a way as to obtain a right-handed coordinate system.

In the vicinity of any given point the equation of the surface will be of the form

$$z+\frac{1}{2}(ax^2+2bxy+cy^2) = 0 \tag{1.03}$$

Since the surface is convex and the z-axis is directed to the convex side we have

$$a > 0 \quad c > 0 \quad ac-b^2 \geqslant 0 \tag{1.04}$$

The equation of the cylindrical surface which separates the region of the geometrical shadow is obtained by eliminating x from (1.01) and (1.02). In our case this equation will be of the form

$$z+\frac{ac-b^2}{2a}y^2 = 0 \tag{1.05}$$

The radius of curvature of the surface in the plane of incidence is equal to

$$R_0 = \frac{1}{a} \tag{1.06}$$

Our problem is to find the electromagnetic field near the surface, at distances (from the surface and from the origin) that are small as compared to the curvature radius R_0.

2. *Simplified Maxwell's Equations*

We suppose the time-dependence of the field components to be of the form $e^{-i\omega t}$ and omit this factor in the following. By k we denote the absolute value of the wave vector

$$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} \tag{2.01}$$

Each of the field components satisfies Helmholtz's equation

$$\Delta\Psi+k^2\Psi = 0 \tag{2.02}$$

where Δ is the Laplace operator. Since we deal with a field due to a plane wave travelling in the direction of the x-axis, we shall separate out the factor e^{ikx} in Ψ and put

$$\Psi = e^{ikx}\Psi^* \tag{2.03}$$

Then Ψ^* will satisfy the equation

$$\frac{\partial^2\Psi^*}{\partial x^2}+\frac{\partial^2\Psi^*}{\partial y^2}+\frac{\partial^2\Psi^*}{\partial z^2}+2ik\frac{\partial\Psi^*}{\partial x} = 0 \tag{2.04}$$

The field components satisfy the Maxwell equations

$$\frac{\partial E_z}{\partial y}-\frac{\partial E_y}{\partial z} = ikH_x, \qquad \text{etc.} \tag{2.05}$$

$$\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z} = -ikE_x, \quad \text{etc.} \tag{2.06}$$

Let us now separate out in each of the field components the factor e^{ikx} and put

$$E_x = E_x^* e^{ikx} \quad \text{etc.}; \qquad H_x = H_x^* e^{ikx} \quad \text{etc.} \tag{2.07}$$

In this way we obtain for the quantities marked by an asterisk the equations:

$$\left.\begin{aligned} \frac{\partial E_z^*}{\partial y} - \frac{\partial E_y^*}{\partial z} &= ikH_x^* \\ \frac{\partial E_x^*}{\partial z} - \frac{\partial E_z^*}{\partial x} - ikE_z^* &= ikH_y^* \\ \frac{\partial E_y^*}{\partial x} - \frac{\partial E_x^*}{\partial y} + ikE_y^* &= ikH_z^* \end{aligned}\right\} \tag{2.08}$$

$$\left.\begin{aligned} \frac{\partial H_z^*}{\partial y} - \frac{\partial H_y^*}{\partial z} &= -ikE_x^* \\ \frac{\partial H_x^*}{\partial z} - \frac{\partial H_z^*}{\partial x} - ikH_z^* &= -ikE_y^* \\ \frac{\partial H_y^*}{\partial x} - \frac{\partial H_x^*}{\partial y} + ikH_y^* &= -ikE_z^* \end{aligned}\right\} \tag{2.09}$$

We shall now introduce an assumption which will be of primary importance in what follows; namely, we suppose that the quantities with asterisks are slowly varying functions of coordinates in the sense that their relative variation in one wave length is small. Also we suppose that the variation of these quantities in the z-direction (normal to the surface) takes place more rapidly than in the x-and y-directions (parallel to the surface).

These assumptions can be stated in the form

$$\frac{\partial \Psi^*}{\partial z} = 0\left(\frac{k}{m}\Psi^*\right); \quad \frac{\partial \Psi^*}{\partial x} = 0\left(\frac{k}{m'}\Psi^*\right); \quad \frac{\partial \Psi^*}{\partial y} = 0\left(\frac{k}{m'}\Psi^*\right) \tag{2.10}$$

m and m' being dimensionless parameters and

$$m' \gg m \gg 1 \tag{2.11}$$

The correctness of these assumptions follows from the fact that the final solution (which is unique) actually satisfies them.

It follows from these assumptions that the second derivatives with respect to x and y in equation (2.04) are small compared to the second deri-

vative with respect to z. Hence this equation takes the form

$$\frac{\partial^2 \Psi^*}{\partial z^2}+2ik\frac{\partial \Psi^*}{\partial x} \doteq 0 \tag{2.12}$$

It follows from (2.12) that m' is of the order of m^2 and we can put

$$m' = m^2 \tag{2.13}$$

The relations (2.10) can now be written in the form

$$\frac{\partial \Psi^*}{\partial x} = 0\left(\frac{k}{m^2}\Psi^*\right); \qquad \frac{\partial \Psi^*}{\partial y} = 0\left(\frac{k}{m^2}\Psi^*\right)$$
$$\frac{\partial \Psi^*}{\partial z} = 0\left(\frac{k}{m}\Psi^*\right) \tag{2.14}$$

From relations (2.14) (which are valid for all the field components) it follows that in equation (2.12) the terms omitted are of the order $1/m^2$ as compared with those written down. Terms of this order of magnitude will always be neglected in the following.

Let us estimate on the basis of (2.14) the order of magnitude of the different terms in equations (2.08) and (2.09). In doing this, we consider H_y^* and H_z^* as the principal quantities to which all the other quantities are to be compared. As to the relative order of magnitude of H_y^* and H_z^*, we shall suppose the order of one of these quantities to differ from that of the other, at the most, by the factor m.

From the first equation (2.09) we get

$$E_x^* = 0\left(\frac{1}{m}H_y^*\right)+0\left(\frac{1}{m^2}H_z^*\right) \tag{2.15}$$

Inserting this estimate into the second equation (2.08) we see that the term $\partial E_x^*/\partial z$ is very small (of the order of $1/m^2$) as compared to the term ikH_y^*. On the other hand, it is seen directly from (2.14) that the term $\partial E_z^*/\partial z$ is of the order $1/m^2$ as compared with ikE_z^*. A term of this order of magnitude must be disregarded. Thus the second equation (2.08) gives simply $E_z^* = -H_y^*$. Similarly the third equation (2.08) gives $E_y^* = H_z^*$, and the first equation (2.08) shows that H_x^* will be of the order

$$H_x^* = 0\left(\frac{1}{m^2}H_y^*\right)+0\left(\frac{1}{m}H_z^*\right) \tag{2.16}$$

These values are also in agreement with equations (2.09).

Hence all the field components may be expressed, neglecting small quantities, in terms of H_y^* and H_z^*. Since these expressions do not involve derivatives with respect to x, they have the same form for the field components without an asterisk, namely:

$$\left.\begin{aligned} E_x &= \frac{i}{k}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right) \\ E_y &= H_z \\ E_z &= -H_y \\ H_z &= \frac{i}{k}\left(\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right) \end{aligned}\right\} \tag{2.17}$$

The last equation can be obtained also directly from div $\mathbf{H}=0$. To these equations we must add the Helmholtz wave equation for each of the field components or an equation of the form (2.12) for the quantities with asterisks.

3. *Simplified Boundary Conditions*

As shown by Leontovich, if the absolute value of the complex inductive capacity of the medium

$$\eta = \varepsilon + i\frac{4\pi\sigma}{ck} \tag{3.01}$$

is great compared to unity, there is no need to consider the field within the medium, but one may take into account the influence of the medium on the field in the air by means of boundary conditions relating the tangential components of this field on the surface of the reflecting body.

Leontovich's conditions (to be more correct, their generalization to the case when the magnetic permeability of the medium is different from unity) can be written in the form of three equations:

$$\left.\begin{aligned} E_x - n_x E_n &= \sqrt{\left(\frac{\mu}{\eta}\right)}(n_y H_z - n_z H_y) \\ E_y - n_y E_n &= \sqrt{\left(\frac{\mu}{\eta}\right)}(n_z H_x - n_x H_z) \\ E_z - n_z E_n &= \sqrt{\left(\frac{\mu}{\eta}\right)}(n_x H_y - n_y H_x) \end{aligned}\right\} \tag{3.02}$$

only two of which are independent. In these equations n_x, n_y, n_z, are com-

ponents of the unit vector of the normal to the surface and E_n has the value

$$E_n = n_x E_x + n_y E_y + n_z E_z \tag{3.03}$$

It can be shown that the conditions (3.02) are valid if the following inequalities are satisfied:

$$|\eta\mu| \gg 1 \tag{3.04}$$

$$kR_0 \sqrt{(|\eta\mu|)} \gg 1 \tag{3.05}$$

where R_0 is the smallest radius of curvature of a normal section of the surface.

In the case of a conductor, in which the displacement current is negligible, these inequalities have the following meaning. According to the first inequality the square of the depth of the skin-effect layer must be small as compared to the square of the wave length in air. According to the second inequality this depth must be small as compared with the radius of curvature of a normal section of the surface.

In the following we put the magnetic permeability equal to unity and transform conditions (3.02) using the relations $E_y = H_z$ and $E_z = -H_y$ obtained above. From (3.02) we get

$$(1-n_x^2)E_x = \left(n_x + \frac{1}{\sqrt{\eta}}\right)(n_y H_z - n_z H_y) \tag{3.06}$$

$$(1-n_x^2)H_x = \left(n_x + \sqrt{\eta}\right)(n_y H_y + n_z H_z) \tag{3.07}$$

Using for H_x the estimate (2.16) and considering, the quantity $\sqrt{\eta}$ to be large (of the order of m or larger), we infer that the left-hand side of (3.07) is small as compared with the separate terms on the right-hand side. Replacing this quantity by zero we obtain instead of (3.07)

$$n_y H_y + n_z H_z = 0 \tag{3.08}$$

Using this relation we obtain from (3.06)

$$n_z E_x = -\left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y \tag{3.09}$$

We can insert in this relation the expression for E_x from the first equation (2.17). Since n_z differs little from unity, we can write

$$\frac{\partial H_y}{\partial z} + ik\left(n_x + \frac{1}{\sqrt{\eta}}\right) H_y = \frac{\partial H_z}{\partial y} \tag{3.10}$$

With the same accuracy equation (3.08) can be written

$$H_z = -n_y H_y \tag{3.11}$$

If the quantities n_x, n_y, $1/m$ are considered small and of the same order of magnitude, and if they are neglected in comparison with unity, the boundary conditions can be simplified still more and can be replaced by the following

$$\frac{\partial H_y}{\partial z}+ik\left(n_x+\frac{1}{\sqrt{\eta}}\right)H_y = 0 \tag{3.12}$$

$$H_z = 0 \tag{3.13}$$

Indeed, since n_y is small, condition (3.11) can be approximately replaced by (3.13); now, the derivative on the right-hand side of (3.10) is taken along the y–axis which is nearly tangential, and thus, if H_z is zero, this derivative will be small and can be neglected to the same approximation.

If boundary conditions (3.12) and (3.13) are used, each of the two components H_y and H_z can be determined separately: each of them satisfies a separate differential equation, a separate boundary condition on the surface of the body and a separate condition at infinity. From these conditions both H_y and H_z are determined uniquely.

In this way the components H_y and H_z can be obtained neglecting quantities of the order $1/m$. To determine these correction terms, we can insert the values of H_y and H_z obtained in the first approximation in the right-hand sides of equations (3.10) and (3.11) and consider these equations as inhomogeneous boundary conditions for the correction terms. These terms must also satisfy the condition at infinity that the part of them which corresponds to the incident wave must have a vanishing amplitude.

Finally, knowing H_y and H_z, we can determine all the other field components from equations (2.17).

4. *Determination of the Field Components H_y and H_z in the First Approximation*

Let us put

$$H_y = H_y^{\circ}\, e^{ikx}\, \Psi^* \tag{4.01}$$

where H_y° is the amplitude of the incident wave at infinity. According to (2.12) and (3.13) the function Ψ^* must satisfy the equation

$$\frac{\partial^2 \Psi^*}{\partial z^2}+2ik\frac{\partial \Psi^*}{\partial x} = 0 \tag{4.02}$$

and the boundary condition

$$\frac{\partial \Psi^*}{\partial z}+ik\left(ax+by+\frac{1}{\sqrt{\eta}}\right)\Psi^* = 0 \tag{4.03}$$

on the surface

$$z+\frac{1}{2}(ax^2+2bxy+cy^2) = 0 \tag{4.04}$$

We have replaced n_x in (4.03) by its approximate value obtained from the equation of the surface.

Suppose that the function Ψ^* depends upon the coordinates x, y, z only through two variables

$$\xi = m(ax+by) \tag{4.05}$$

$$\zeta = 2am^2\left[z+\frac{1}{2}(ax^2+2bxy+cy^2)\right] \tag{4.06}$$

where m is a large parameter which will be defined below. The scales of the quantities ξ and ζ are chosen in such a way that equation (1.05) (giving the shadow boundary in space) takes the form

$$\zeta = \xi^2 \tag{4.07}$$

The values of the variable ζ can be only non-negative and those of the variable ξ can be both positive and negative.

In the illuminated region of the space we have $\xi < \sqrt{\zeta}$ and in the shaded one $\xi > \sqrt{\zeta}$, where the square root is taken with a positive sign.

Calculating the derivatives we obtain

$$\frac{\partial\Psi^*}{\partial x} = ma\left(\frac{\partial\Psi^*}{\partial\xi}+2\xi\frac{\partial\Psi^*}{\partial\zeta}\right) \tag{4.08}$$

$$\frac{\partial\Psi^*}{\partial z} = 2m^2a\frac{\partial\Psi^*}{\partial\zeta} \tag{4.09}$$

and equation (4.02) takes the form:

$$\frac{\partial^2\Psi^*}{\partial\zeta^2}+i\frac{k}{2m^3a}\left(\frac{\partial\Psi^*}{\partial\xi}+2\xi\frac{\partial\Psi^*}{\partial\zeta}\right) = 0 \tag{4.10}$$

We now choose the parameter m in such a way as to make the coefficient in this equation equal to unity

$$m = \sqrt[3]{\left(\frac{k}{2a}\right)} = \sqrt[3]{\left(\frac{kR_0}{2}\right)} \tag{4.11}$$

Since we consider the wave length to be very small as compared with the radius of curvature of the surface, the value of our parameter m

will actually be large. The expressions for the derivatives can now be written in the following form:

$$\frac{\partial \Psi^*}{\partial x} = \frac{k}{2m^2}\left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta}\right) \tag{4.12}$$

$$\frac{\partial \Psi^*}{\partial z} = \frac{k}{m}\,\frac{\partial \Psi^*}{\partial \zeta} \tag{4.13}$$

It is seen from these equations that the estimates (2.14) will be valid, provided the derivatives of Ψ^* with respect to ξ and to ζ are of the same order as Ψ^* itself.

Equation (4.10) takes the form

$$\frac{\partial^2 \Psi^*}{\partial \zeta^2} + i\left(\frac{\partial \Psi^*}{\partial \xi} + 2\xi \frac{\partial \Psi^*}{\partial \zeta}\right) = 0 \tag{4.14}$$

The boundary condition (4.03) becomes

$$\frac{\partial \Psi^*}{\partial \zeta} + i\xi \Psi^* + q\Psi = 0^* \tag{4.15}$$

where we have put for brevity:

$$q = \frac{im}{\sqrt{\eta}} = \frac{i}{\sqrt{\eta}}\sqrt[3]{\left(\frac{k}{2a}\right)} \tag{4.16}$$

The quantity q will be, in general, finite, but can be also small (for a very good conductor) or large (for an almost plane surface).

The condition at infinity for Ψ^* consists of the following. In the illuminated region that part of Ψ^*, the phase of which vanishes, must have an amplitude equal to unity.

To simplify the differential equation we put

$$\Psi^* = \mathrm{e}^{-i\xi\zeta + \frac{i}{3}\xi^3} \cdot V \tag{4.17}$$

Then the equation and the boundary condition for V will be

$$\frac{\partial^2 V}{\partial \zeta^2} + i\frac{\partial V}{\partial \xi} + \zeta V = 0 \tag{4.18}$$

$$\frac{\partial V}{\partial \zeta} + qV = 0; \qquad (\text{for } \zeta = 0) \tag{4.19}$$

The condition at infinity (large negative values of ξ) becomes

$$V = \mathrm{e}^{i\xi\zeta - \frac{i}{3}\xi^3} - V^* \tag{4.20}$$

where V^* corresponds to the reflected wave. We denote the phase of the first term in (4.20) by φ

$$\varphi = \xi\zeta - \frac{1}{3}\xi^3 \tag{4.21}$$

and the phase of V^* by φ^*. The phase φ^* can be determined by calculating from geometrical considerations the phase difference $\varphi^* - \varphi$ between the reflected and the incident wave and by using the known value (4.21) of the phase φ.

It can be shown that the phase φ^* so determined is equal to the extremum value of the function

$$\varphi^* = t\xi + \frac{2}{3}(\zeta - t)^{3/2} - \frac{4}{3}(-t)^{3/2} \tag{4.22}$$

i.e. equal to φ^* for that value of t, for which $\partial\varphi^*/\partial t = 0$. Similarly the given phase (4.21) is equal to the extremum value of the function

$$\varphi = t\xi - \frac{2}{3}(\zeta - t)^{3/2} \tag{4.23}$$

We omit the derivation, since it is rather cumbersome and since the result can be obtained in a purely analytical way from the final form of the solution (see Section 6).

Equation (4.18) coincides with that which occurs in the problem of diffraction of radio waves around the earth's surface. This equation (with different conditions at infinity) is investigated in Chapter 11.

Equation (4.18) admits particular solutions of the form

$$V = e^{i\xi t} w(t - \zeta) \tag{4.24}$$

where $w(t)$ is a solution of the ordinary differential equation of the second order

$$w''(t) = tw(t) \tag{4.25}$$

We shall need both solutions of equation (4.25). As one of these solutions we take the function

$$w_1(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_1} e^{zt - \frac{1}{3}z^3} dz \tag{4.26}$$

where the contour Γ_1 goes from infinity to the origin along the line arc $z = -2\pi/3$ and then returns to infinity along the line arc $z = 0$ (along

the positive real axis). Another (linearly independent) solution is the function

$$w_2(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma_2} e^{zt-\frac{1}{3}z^3}\, dz \tag{4.27}$$

where the contour Γ_2 is an image of the contour Γ_1 in the real axis of the z-plane. For real values of t the functions $w_1(t)$ and $w_2(t)$ are complex-conjugates. We shall have

$$\left.\begin{aligned} w_1(t) &= u(t)+iv(t) \\ w_2(t) &= u(t)-iv(t) \end{aligned}\right\} \tag{4.28}$$

For real functions $u(t)$ and $v(t)$ and their derivatives, extensive four-figure tables (range from $t=-9{\cdot}00$ to $t=+9{\cdot}00$, interval $0{\cdot}02$) have been computed (see Appendix on p. 379).

The asymptotic expression for $w_1(t)$, valid for large negative values of t (and also in a certain sector in the plane of the complex variable t) has the form

$$w_1(t) = (-t)^{-\frac{1}{4}} \exp\left(i\frac{2}{3}(-t)^{3/2}+i\frac{\pi}{4}\right) \tag{4.29}$$

Similarly

$$w_2(t) = (-t)^{-\frac{1}{4}} \exp\left(-i\frac{2}{3}(-t)^{3/2}-i\frac{\pi}{4}\right) \tag{4.30}$$

From (4.23) and (4.30) we see that the phase of the expression

$$e^{i\xi t}\, w_2(t-\zeta) \tag{4.31}$$

is just equal to φ; and we know that the extremum of φ gives the phase of the incident wave. Therefore, we can expect that the integration of the function (4.31) along a contour which passes near the point of the extremum of the phase, gives an expression, the phase of which is equal to that of the incident wave (4.21). In fact, making use of the relations:

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{xp}\, v(p)\, dp = e^{\frac{1}{3}x^3} \qquad (\operatorname{Re} x \geqslant 0) \tag{4.32}$$

$$w_2\left(p\, e^{i\frac{2\pi}{3}}\right) = 2\, e^{-i\pi/6}\, v(p) \tag{4.33}$$

the following equality can be proved:

$$e^{i\xi\zeta-\frac{i}{3}\xi^3} = \frac{i}{2\sqrt{\pi}} \int_C e^{it\xi}\, w_2(t-\zeta)\, dt \tag{4.34}$$

where the contour C runs along the line arc $t=2\pi/3$ from infinity to the origin and along the line arc $t = -\pi/3$ from the origin to infinity.

On the other hand, if the function $f(t)$ is such that its phase for large negative values of t is equal to $-(4/3)(-t)^{3/2}$ then the phase of the expression

$$e^{i\xi t}\, w_1(t-\zeta)f(t) \tag{4.35}$$

is equal to φ^* (in formula (4.22)). Hence, integrating expression (4.35) along a contour, which passes in the vicinity of the point of the extremum of the phase, we obtain an expression which has a phase equal to that of the reflected wave.

From these considerations it follows that we may seek the expression for V in the form

$$V = \frac{i}{2\sqrt{\pi}}\int_C e^{i\xi t}\left[w_2(t-\zeta)-f(t)w_1(t-\zeta)\right]dt \tag{4.36}$$

This expression satisfies equation (4.18) and the condition at infinity (4.20). To satisfy also the boundary condition (4.19) we have to determine the function $f(t)$ from the relation

$$w_2'(t)-qw_2(t) = f(t)\left(w_1'(t)-qw_1(t)\right) \tag{4.37}$$

whence

$$f(t) = \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} \tag{4.38}$$

It is not difficult to see from (4.29) and (4.30) that the function $f(t)$ so obtained has the correct phase.

We, finally, obtain

$$V = \frac{i}{2\sqrt{\pi}}\int_C e^{i\xi t}\left[w_2(t-\zeta)-\frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\,w_1(t-\zeta)\right]dt \tag{4.39}$$

With this value of V the expression

$$H_y = H_y^0\, e^{ikx}\cdot e^{-i\xi\zeta+\frac{i}{3}\xi^3}\cdot V \tag{4.40}$$

gives the y-component of the magnetic field.

Using the relation

$$w_1'(t)w_2(t)-w_2'(t)w_1(t) = -2i \tag{4.41}$$

it is easy to verify that at $\zeta=0$ (on the surface of the body) the expression (4.39) for V becomes

$$V = \frac{1}{\sqrt{\pi}}\int_C e^{i\xi t}\,\frac{dt}{w_1'(t)-qw_1(t)} \tag{4.42}$$

Inserting this in (4.40), we arrive at the following conclusion. The tangential components H_{tg} of the magnetic field on the surface of the body are equal to their values H_{tg}^{ex} for the external field, multiplied by a certain universal function of the reduced distance ξ from the shadow boundary and of the parameter q (the latter depends upon the wave length and the properties of the body). We have

$$H_{tg} = H_{tg}^{ex} G(\xi, q) \tag{4.43}$$

where

$$G(\xi, q) = e^{-\frac{i}{3}\xi^3} \frac{1}{\sqrt{\pi}} \int_C e^{i\xi t} \frac{dt}{w_1'(t) - q w_1(t)} \tag{4.44}$$

This result is in agreement with that obtained in Chapter 2 by a wholly different method and represents a generalization to the case of finite electrical conductivity of the body.

For a perfect conductor $q=0$ we have

$$G(\xi, 0) = G(\xi) \tag{4.45}$$

where $G(\xi)$ is a function tabulated in Chapter 2.

We note that the quantity V determined by (4.42) occurred also in our solution of the problem of the propagation of radio waves around the earth's surface (it was denoted there by $V_1(\xi, q)$) (Chapter 10).

We have still to determine the component of the magnetic field H_z from the differential equation and from the boundary condition $H_z = 0$ on the surface of the body.

Let us put

$$H_z = H_z^{\circ} e^{ikx} \Phi^* \tag{4.46}$$

where H_z° is the amplitude of the incident wave at infinity.

The function Φ^* must satisfy the equation

$$\frac{\partial^2 \Phi^*}{\partial z^2} + 2ik \frac{\partial \Phi^*}{\partial x} = 0 \tag{4.47}$$

and the boundary condition

$$\Phi^* = 0 \quad \text{(on the surface of the body).} \tag{4.48}$$

The condition at infinity will be the same as the corresponding condition for Ψ^*.

We assume that Φ^* depends on the same variables ξ, ζ as Ψ^* and make the substitution

$$\Phi^* = e^{-i\xi\zeta + \frac{i}{3}\xi^3} U \tag{4.49}$$

Since Φ^* satisfies the same equation as Ψ^*, the equation for U coincides with equation (4.18) for V. In order to determine U we have, therefore, the equation

$$\frac{\partial^2 U}{\partial \zeta^2}+i\frac{\partial U}{\partial \xi}+\zeta U = 0 \tag{4.50}$$

the boundary condition

$$U = 0 \qquad \text{for} \qquad \zeta = 0 \tag{4.51}$$

and the condition at infinity

$$U = \mathrm{e}^{i\xi\zeta-\frac{i}{3}\xi^3} - U^* \tag{4.52}$$

where U^* corresponds to the reflected wave.

If we assume for U an expression of the form (4.36), the function $f(t)$ will be determined from equation (4.51), and we obtain

$$U = \frac{i}{2\sqrt{\pi}}\int_C \mathrm{e}^{i\xi t}\left[w_2(t-\zeta)-\frac{w_2(t)}{w_1(t)}w_1(t-\zeta)\right]\mathrm{d}t \tag{4.53}$$

Inserting (4.53) and (4.49) into (4.46), we obtain the value of H_z in the first approximation. We see that the normal component of the magnetic field on the surface of the body is equal to its value for the external field, multiplied by a universal function Φ^*.

5. *Determination of the Field in the Second Approximation*

In the second approximation the boundary conditions for H_y and H_z will have the form (3.10) and (3.11). To satisfy them, we put

$$H_y = H_y^\circ \mathrm{e}^{ikx}\Psi^* + \frac{1}{m}H_z^\circ \mathrm{e}^{ikx}Q^* \tag{5.01}$$

$$H_z = \frac{1}{m}H_y^\circ \mathrm{e}^{ikx}P^* + H_z^\circ \mathrm{e}^{ikx}\Phi^* \tag{5.02}$$

where m is the large parameter introduced above and the functions P^* and Q^* are of the same order of magnitude as the functions Ψ^* and Φ^* previously obtained. All four functions P^*, Q^*, Ψ^*, Φ^* satisfy differential equations of the form (4.02) or (4.14), and the functions Ψ^* and Φ^* satisfy the boundary conditions

$$\frac{\partial \Psi^*}{\partial \zeta}+(i\xi+q)\Psi^* = 0; \qquad \Phi^* = 0 \tag{5.03}$$

on the surface of the body.

When inserting the values of H_y and H_z in the boundary conditions (3.10) and (3.11) we must use on the left-hand side the full expressions (5.01) and (5.02), while on the right-hand side we may omit the correction terms involving P^* and Q^*. Taking account of the boundary conditions (5.03) for Ψ^* and Φ^*, we obtain

$$P^* = -m(bx+cy)\Psi^* \tag{5.04}$$

$$\frac{\partial Q^*}{\partial \zeta}+(i\xi+q)Q^* = m(bx+cy)\frac{\partial \Phi^*}{\partial \zeta} \tag{5.05}$$

If we use the identity

$$bx+cy = \frac{b}{a}(ax+by)+\frac{ac-b^2}{a}y \tag{5.06}$$

or

$$m(bx+cy) = \frac{b}{a}\xi+\frac{ac-b^2}{a}\cdot my \tag{5.07}$$

we can write instead of (5.04) and (5.05)

$$P^* = -\frac{b}{a}\xi\Psi^*-\frac{ac-b^2}{a}my\Psi^* \tag{5.08}$$

$$\frac{\partial Q^*}{\partial \zeta}+(i\xi+q)Q^* = \left(\frac{b}{a}\xi+\frac{ac-b^2}{a}my\right)\frac{\partial \Phi^*}{\partial \zeta} \tag{5.09}$$

But according to the boundary condition (5.03) for Ψ^* we have on the surface of the body

$$\xi\Psi^* = i\left(\frac{\partial \Psi^*}{\partial \zeta}+q\Psi^*\right) \tag{5.10}$$

Equation (5.08) can therefore be written as

$$P^* = -\frac{ib}{a}\frac{\partial \Psi^*}{\partial \zeta}-\left(\frac{ibq}{a}+\frac{ac-b^2}{a}\cdot my\right)\Psi^* \tag{5.11}$$

This equation must hold on the surface of the body, but its right-hand side has a definite meaning and can be used in a region near the body as well. It is easy to see that the expression (5.11) satisfies there the same parabolic equation as Ψ^*. Indeed, if we write the parabolic equation in the form (4.14), it will be evident that it is satisfied also by $\partial\Psi^*/\partial\zeta$. But since this equation may be also written in the form (4.02) not involving differentiations with respect to y, it is satisfied also by the second term in (5.11).

In addition to the differential equation the quantity P^* must satisfy the condition at infinity, according to which the amplitude of the term corresponding to the incident wave must vanish there. This condition is not satisfied by the expression (5.11), but this expression can be easily modified, without violating the boundary condition, so as to satisfy the condition at infinity as well. For this purpose it is sufficient to add to (5.11) a term proportional to Φ^*. The expression

$$P^* = -\frac{ib}{a}\frac{\partial \Psi^*}{\partial \zeta}+\left(\frac{ibq}{a}+\frac{ac-b^2}{a}my\right)(\Phi^*-\Psi^*) \tag{5.12}$$

actually satisfies all the condition imposed: the parabolic differential equation, the boundary condition on the surface of the body and the condition at infinity. Inserted into (5.02) it gives therefore the required correction term in the field component H_z.

We now proceed to the determination of Q^*. Neglecting for the moment the condition at infinity, we can try the following expression for Q^*:

$$Q^* = A\frac{\partial \Phi^*}{\partial \zeta}+B\Phi^* \tag{5.13}$$

Here, A is a constant and B may be a function of y. Using the differential equation for Φ^*, we obtain for the derivative with respect to ζ the expression

$$\frac{\partial Q^*}{\partial \zeta} = -2iA\xi\frac{\partial \Phi^*}{\partial \zeta}-iA\frac{\partial \Phi^*}{\partial \xi}+B\frac{\partial \Phi^*}{\partial \zeta} \tag{5.14}$$

and since on the surface of the body we have $\Phi^*=0$ and $\partial\Phi^*/\partial\xi=0$, the left-hand side of (5.09) will be equal to

$$\frac{\partial Q^*}{\partial \zeta}+(i\xi+q)Q^* = (-iA\xi+Aq+B)\frac{\partial \Phi^*}{\partial \zeta} \tag{5.15}$$

Comparing this with (5.09), we find the values of the quantities A and B, namely

$$A = \frac{ib}{a}; \qquad B = -\frac{ib}{a}q+\frac{ac-b^2}{a}my \tag{5.16}$$

Therefore, the expression

$$Q^* = \frac{ib}{a}\frac{\partial \Phi^*}{\partial \zeta}+\left(-\frac{ib}{a}q+\frac{ac-b^2}{a}my\right)\Phi^* \tag{5.17}$$

satisfies the differential equation and the condition on the surface of the

body. To satisfy the condition at infinity, we observe that the homogeneous boundary condition, corresponding to the inhomogeneous condition (5.09) for Q^*, is satisfied by Ψ^*. Without violating the boundary condition (5.09) we can add, therefore, to Q^* a quantity proportional to Ψ^* (the proportionality factor may depend on y). The expression for Q^* so constructed, namely

$$Q^* = \frac{ib}{a}\frac{\partial \Phi^*}{\partial \zeta} + \left(-\frac{ib}{a}q + \frac{ac-b^2}{a}my\right)(\Phi^* - \Psi^*) \tag{5.18}$$

will satisfy all the conditions imposed and gives the correction term in the expression (5.01) for the field component H_y.

We observe that the expression (5.12) and (5.18) obtained for P^* and Q^* have a certain symmetry and go over into one another by the interchange

$$\Phi^* \to -\Psi^*, \quad \Psi^* \to -\Phi^*, \quad q \to -q \tag{5.19}$$

Introducing these expressions into (5.01) and (5.02), we obtain the field components H_y and H_z with the correction terms. The other field components are obtained from the simplified Maxwell equations (2.17). We have

$$E_y = H_z; \qquad E_z = -H_y \tag{5.20}$$

and neglecting small terms

$$E_x = -\frac{i}{m} H_y^\circ e^{ikx} \frac{\partial \Psi^*}{\partial \zeta}; \qquad H_x = \frac{i}{m} H_z^\circ e^{ikx} \frac{\partial \Phi^*}{\partial \zeta} \tag{5.21}$$

The field components E_x and H_x are thus of the same order as the correction terms in the other field components.

The determination of the field components is now complete.

6. *The Field in the Illuminated Region*

In order to investigate the field in the illuminated region we have to deduce for the functions U and V given by (4.53) and (4.39) asymptotic expressions, valid for large negative values of ξ.

We put according to (4.21)

$$\varphi = \xi\zeta - \frac{1}{3}\xi^3 \tag{6.01}$$

Then we have

$$U = e^{i\varphi} - U^* \tag{6.02}$$

$$V = e^{i\varphi} - V^* \tag{6.03}$$

where

$$U^* = \frac{i}{2\sqrt{\pi}} \int_C e^{i\xi t} \frac{w_2(t)}{w_1(t)} w_1(t-\zeta)\, dt \tag{6.04}$$

$$V^* = \frac{i}{2\sqrt{\pi}} \int_c e^{i\xi t} \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} w_1(t-\zeta)\, dt \tag{6.05}$$

The phase of the integrands in U^* and V^* is equal to the expression

$$\varphi^* = t\xi + \frac{2}{3}(\zeta - t)^{3/2} - \frac{4}{3}(-t)^{3/2} \tag{6.06}$$

which was considered above (formula (4.22)).

In the point of the extremum of the phase we have

$$\sqrt{-t} = \frac{1}{3}\sigma - \frac{2}{3}\xi \tag{6.07}$$

$$\sqrt{(\zeta - t)} = \frac{2}{3}\sigma - \frac{1}{3}\xi \tag{6.08}$$

where we have put for brevity

$$\sigma = \sqrt{(\xi^2 + 3\zeta)} \tag{6.09}$$

the root being taken positive.

The extremum value of the phase is equal to

$$\varphi^* = \frac{1}{27}(4\sigma^3 - 3\sigma^2\xi - 2\xi^3) \tag{6.10}$$

In the following we shall always use the symbol φ^* to denote this extremum value. Applying the method of stationary phase we obtain for U^* the asymptotic expression

$$U^* = e^{i\varphi^*} \sqrt{\left(\frac{1}{3} - \frac{2}{3}\frac{\xi}{\sigma}\right)} \tag{6.11}$$

The integrand in V^* differs from that in U^* by a slowly varying factor, which is for large negative values of t approximately equal to

$$\frac{(w_2'/w_2) - q}{(w_1'/w_1) - q} = \frac{q - i\sqrt{-t}}{q + i\sqrt{-t}} \tag{6.12}$$

Therefore the asymptotic value of V^* will differ from that of U^* by the

factor (6.12) taken at the extremum point. Hence we have

$$V^* = e^{i\varphi^*}\sqrt{\left(\frac{1}{3}-\frac{2\xi}{3\sigma}\right)}\,\frac{q-\frac{i}{3}(\sigma-2\xi)}{q+\frac{i}{3}(\sigma-2\xi)} \tag{6.13}$$

Let us elucidate the geometrical meaning of the formulae obtained.

We consider the ray, which goes through the point x, y, z after reflection. Determining the coordinates x_0, y_0, z_0 of the point on the surface, where the reflection took place, we obtain the following approximate formulae, valid for glancing incidence:

$$x_0 = x-s; \qquad y_0 = y \tag{6.14}$$

where

$$s = \frac{\sigma+\xi}{3am} \tag{6.15}$$

Geometrically s is the length of the path, travelled by the ray after reflection. The cosine of the incidence angle is equal to

$$\cos\theta = -(ax_0+by_0) = \frac{1}{3m}(\sigma-2\xi) \tag{6.16}$$

The exact value of the difference x_0-x+s is

$$x_0-x+s = 2s\cos^2\theta \tag{6.17}$$

The phase difference of the reflected and the incident wave is proportional to this quantity. We have

$$\varphi^*-\varphi = k(x_0-x+s) = 2ks\cos^2\theta \tag{6.18}$$

Inserting in (6.18) the values of s and of $\cos\theta$ from (6.15) and (6.16) and using (4.11) we obtain

$$\varphi^*-\varphi = \frac{4}{27}(\sigma+\xi)(\sigma-2\xi)^2 \tag{6.19}$$

It is easy to verify that (6.19) is equal to the difference of the quantities (6.10) and (6.01).

Hence the phase difference of the two terms in (6.02) and (6.03) is in agreement with the results obtained from geometrical optics.

Consider now the amplitude of the reflected wave.

Inserting (6.11) into (6.02) we have

$$U = e^{i\varphi} - e^{i\varphi^*} \sqrt{\left(\frac{1}{3} - \frac{2\xi}{3\sigma}\right)} \tag{6.20}$$

Using the expression (4.16) for q and the value (6.16) for $\cos\theta$ and inserting (6.13) into (6.03) we obtain

$$V = e^{i\varphi} - e^{i\varphi^*} \sqrt{\left(\frac{1}{3} - \frac{2\xi}{3\sigma}\right)} \frac{1-\cos\theta\sqrt{\eta}}{1+\cos\theta\sqrt{\eta}} \tag{6.21}$$

The function U corresponds to the case when the polarization of the incident wave is such that the electric vector is perpendicular to the plane of incidence. The function V corresponds to the case of an electric vector parallel to the plane of incidence. It is easy to see that in both cases our formulae give the correct values of the Fresnel coefficients.

7. *Conclusion*

The formulae obtained above give immediately the field in the vicinity of any point situated on the surface of a conducting body on the boundary of the geometrical shadow. Since this point may be chosen in an arbitrary way, our formulae give also the field in a certain ring-shaped region adjacent to the closed line, which represents the boundary of the geometrical shadow on the surface (penumbra region). Consider now the field outside this region, but still near the surface (at distances from the surface, that are small as compared with its radius of curvature). In the shaded part of this region we may put the field amplitude equal to zero. Indeed, the solution obtained decreases exponentially as the distance from the shadow boundary increases, and, if the quantity $\xi - \sqrt{\zeta}$ is positive and large, this solution can be considered to be zero. We thus obtain a continuous transition to complete shadow. Let us now consider the illuminated region. In Section 6 we have seen that in the remote part of the illuminated region our formulae give a field which coincides with that obtained from the Fresnel formulae. Hence it follows that if we use our formulae in the penumbra region and calculate the field with the help of Fresnel's formulae in the illuminated one, we shall obtain a continuous transition from penumbra to light.

In this way our formulae permit us to determine the field on and near the whole surface of the body (within a certain layer). Particularly, they give the current distribution, induced by an incident plane wave on the surface of the body. But if the current distribution is known, the field

of the scattered wave can be determined in the whole space (also at large distances from the body) by applying well-known formulae for the vector-potential due to given currents.

As a final result our formulae give a complete (though approximate) solution of the problem of diffraction of a plane wave by a conducting convex body of arbitrary shape.

CHAPTER 6

FRESNEL'S REFLECTION LAWS AND DIFFRACTION LAWS†

IN 1821, the French scientist Fresnel established formulae determining the intensity and direction of oscillations in reflected and refracted rays of light incident on the plane surface of a transparent body.

Fresnel obtained his formulae on the basis of the elastic theory of light. He assumed transverse oscillations of the elastic medium (ether) and was obliged to introduce special hypotheses on the elasticity and density of the ether in media which differed in index of refraction. This derivation does not agree with the modern view on the nature of light and has only historical interest at the present time. However, the formulae themselves were justified brilliantly by experiment and served, later, as touchstones for the verification of every theory of light which was proposed.

In 1865, the electro-magnetic theory of light, created by Maxwell, appeared, which stood this test and, moreover, gave an explanation of a wide range of phenomena including those which were detected many years later such as: radiowaves (Hertz, Popov), light pressure (Lebedev) and many others.

The Fresnel reflection laws can be deduced from the Maxwell equations and the appropriate boundary conditions without any additional hypotheses, and it appears that the transverse oscillations analysed by Fresnel must be understood as the oscillations of the electric vector.

The Fresnel laws are applicable not only to light but also to electromagnetic oscillations of any frequency, including radiowaves. Moreover they can be easily generalized to the case where the waves fall on the plane surface of an *absorbing* body. The Fresnel formulae retain their form, with this sole difference: the refractive index n must be replaced by a complex quantity, namely, the square root of the complex dielectric constant of the medium.

The Fresnel formulae allow the direct expression of the amplitudes of

† Fock, 1948.

the electromagnetic field of the *reflected* wave through the field amplitudes of the *incident* wave (their values on the reflecting surface). If a plane wave falls on the surface, and if the reflecting surface itself is plane, then the field amplitudes of the reflected wave at a certain distance from the surface will be the same as on the surface itself; only the phase will depend on the distance from the surface. If the reflecting surface is convex, then the incident parallel beam of rays becomes divergent after reflection. In such a case, when calculating the reflected-wave amplitudes at a given distance from the point of reflection, it is necessary to introduce into the amplitude a correction factor which would take into account the spreading of the beam after reflection. This factor can be found from purely geometrical considerations.

The electro-magnetic wave reflection laws admit a very simple and convenient approximate formulation in terms of the Fresnel formulae. It is a much less satisfactory matter in the case of the approximate formulation of the *diffraction* laws; diffraction being understood as the tendency of the wave to bend round an obstacle and to penetrate into the region of the geometrical shadow. Until very recently all the known approximate methods referred to the case of wave diffraction from an obstacle with *sharp* edges, for example, from an opaque screen with holes. These methods are in essence refinements of the Huygens principle. The principal step in this direction was made by Fresnel himself. According to the Huygens principle in the Fresnel formulation, that part of the light wave which is covered by the screen does not act at all, but the uncovered regions act just as though there were no screen at all. A further improvement was made in 1882 by Kirchhoff who proposed a formula for the amplitude of light waves outside a screen. The Kirchhoff formula is a very flexible and convenient means of solving approximately the problem of diffraction from a screen with sharp edges but it does not take into account the influence of the screen material and neglects the boundary conditions for the field which follow from the Maxwell equations.

The next substantial step in the solution of the diffraction problem from a screen with sharp edges was made when rigorous solutions of the Maxwell equations for certain particular cases (half-plane, wedge) were found. Here the work of Sommerfeld should be mentioned and also the work of S. L. Sobolev and V. I. Smirnov, who approached the problem from a new point of view (non-stationary process). The interesting problems of the plane and cylindrical waveguides with open ends (where the diffracted wave can bend backwards) were solved recently by L. A. Wainstein.

In contrast to the problem of diffraction from bodies with sharp edges (screens and diaphragms), no general approximate methods or approximate formulae (similar to the Kirchhoff formulae) have been proposed for the solution of the problem of diffraction from bodies with continuously varying curvature. In order to find the field due to the diffraction of the incident wave, one had to solve the Maxwell equations for each particular case with proper limiting conditions for each separate case; this is a very difficult mathematical problem.

The Fresnel reflection formulae represent an integral law in the sense that their use does not require the solution of differential equations; these formulae give explicit expressions for the reflected wave amplitude. Not only was the form of the appropriate integral law not known for the phenomenon of diffraction from bodies of arbitrary shape, but even the fact of the existence of such a law was not established. In other words, the possibility of writing explicit expressions for the field amplitude of waves bending around a body under any general assumptions of the electrical properties of the material of the body and on the shape of its surface was not established.

This gap was filled to some extent in our investigations on the diffraction of plane waves around the surface of a convex conducting body of arbitrary shape.

The assumption that the material of the body is a good conductor is essential because it affords the possibility of using the simplified boundary conditions for the field established by M. A. Leontovich.

Considering the field near the surface of the body (at distances which are small in comparison with the radii of curvature of the surface), we established that in the penumbra region this field has a local character. This means that, for a given wavelength, amplitude and polarization of the incident wave, the field in the penumbra region depends only on the shape and properties of the body near the given point, and is expressible in terms of certain universal functions which can be tabulated once and for all. These expressions thus represent general diffraction laws.

Our formulae for the field can be considered as a generalization of the Fresnel formulae – a generalization which includes both the reflection and the diffraction laws.

Let us move, mentally, along the surface of a body from its illuminated side to the shadow. On the illuminated side the incident and the reflected waves can be distingushed from one another and the latter will be described well by the Fresnel formulae. Near the geometrical boundary of the shadow, in the region of very oblique (glancing) incidence of the ray, both

waves become inseparable from each other so that only the consideration of the resultant field has any meaning. Here, our formulae are to be used while the Fresnel formulae become inapplicable. Beyond the geometrical boundary of the shadow we do not have waves of more or less constant amplitude but we have a damped wave, i.e., a wave with an amplitude decreasing exponentially as the distance from the geometrical boundary of the shadow increases. Here, the diffraction phenomenon in its proper sense takes place, the diffraction law being expressed by our formulae.

From the above, it is clear that a region exists (namely the region of oblique ray incidence) where both our diffraction formulae and the Fresnel formulae are valid simultaneously. Evidently, in this region one formula must go over into the other.

In the following, we will give the Fresnel formulae for the electromagnetic field and find their generalization which allows us to take into account the broadening of the beam after its reflection from a convex body. We will then write the diffraction formulae and consider their limiting cases; in particular we will show how they transform into the Fresnel formulae in the region of oblique incidence.

1. *Fresnel Reflection Laws*

Let us denote the amplitudes of the electric and magnetic vectors of an incident wave at a given point on the surface of the body by $\mathbf{E}^\circ(E_x^\circ, E_y^\circ, E_z^\circ)$ and $\mathbf{H}^\circ(H_x^\circ, H_y^\circ, H_z^\circ)$. Let us denote the corresponding quantities for the reflected wave by $\mathbf{E}^*(E_x^*, E_y^*, E_z^*)$ and $\mathbf{H}^*(H_x^*, H_y^*, H_z^*)$. Let $\mathbf{a}(a_x, a_y, a_z)$ be, further, the unit vector in the direction of the incident ray, $\mathbf{a}^*(a_x^*, a_y^*, a_z^*)$ the unit vector in the direction of the reflected ray and $\mathbf{n}(n_x, n_y, n_z)$ the unit vector of the normal to the surface of the body at the point of incidence. According to the reflection law, the quantities $\mathbf{a}^*$, $\mathbf{a}$ and $\mathbf{n}$ are connected by the relation

$$\mathbf{a}^* = \mathbf{a} - 2\mathbf{n}(\mathbf{a}\cdot\mathbf{n}) \tag{1.01}$$

and

$$\mathbf{a}^*\cdot\mathbf{n} = -\mathbf{a}\cdot\mathbf{n} = \cos\vartheta \tag{1.02}$$

where ϑ is the angle of incidence. The vectors $\mathbf{a}$ and $\mathbf{a}^*$ are proportional to the gradient of the phases of the incident and the reflected waves. Considering the amplitude to be a quantity which varies slowly as compared to the phase, we obtain from the Maxwell equations for vacuum

$$[\mathbf{a}\times\mathbf{E}^\circ] = \mathbf{H}^\circ; \qquad \mathbf{a}\cdot\mathbf{E}^\circ = 0 \tag{1.03}$$

from which

$$[\mathbf{a}\times\mathbf{H}^\circ] = -\mathbf{E}^\circ; \qquad \mathbf{a}\cdot\mathbf{H}^\circ = 0 \tag{1.04}$$

and similarly for the reflected wave:

$$[\mathbf{a}^* \times \mathbf{E}^*] = \mathbf{H}^*; \qquad \mathbf{a}^* \cdot \mathbf{E}^* = 0 \tag{1.05}$$

$$[\mathbf{a}^* \times \mathbf{H}^*] = -\mathbf{E}^*; \qquad \mathbf{a} \cdot \mathbf{H}^* = 0 \tag{1.06}$$

Let us denote the magnetic permability by μ, the complex dielectric constant of the substance of the reflecting body by

$$\eta = \varepsilon + i\frac{4\pi\sigma}{\omega} \tag{1.07}$$

and let us introduce the Fresnel coefficients

$$N = \frac{\eta \cos\vartheta - \sqrt{(\mu\eta - \sin^2\vartheta)}}{\eta \cos\vartheta + \sqrt{(\mu\eta - \sin^2\vartheta)}} \tag{1.08}$$

$$M = \frac{\mu \cos\vartheta - \sqrt{(\mu\eta - \sin^2\vartheta)}}{\mu \cos\vartheta + \sqrt{(\mu\eta - \sin^2\vartheta)}} \tag{1.09}$$

Then the Fresnel formulae establishing the relation between the amplitudes of the incident and the reflected wave can be written in the form

$$(\mathbf{n} \cdot \mathbf{E}^*) = N(\mathbf{n} \cdot \mathbf{E}^\circ) \tag{1.10}$$

$$(\mathbf{n} \cdot \mathbf{H}^*) = M(\mathbf{n} \cdot \mathbf{H}^\circ) \tag{1.11}$$

The amplitudes of the transmitted wave (penetrating the substance of the body) are not of interest for our purpose and we will not write the corresponding formulae.

Equations (1.05), (1.10) and (1.11) can be solved with respect to the vectors $\mathbf{E}^*$ and $\mathbf{H}^*$. Introducing the notation

$$\mathbf{n} \cdot \mathbf{E}^\circ = E_n^\circ; \qquad \mathbf{n} \cdot \mathbf{H}^\circ = H_n^\circ \tag{1.12}$$

and expressing $\mathbf{a}^*$ in terms of $\mathbf{a}$ according to (1.01), we will have:

$$\sin^2\vartheta\, \mathbf{E}^* = -NE_n^\circ(\mathbf{n} \cos 2\vartheta + \mathbf{a} \cos\vartheta) + MH_n^\circ[\mathbf{n} \times \mathbf{a}] \tag{1.13}$$

$$\sin^2\vartheta\, \mathbf{H}^* = -MH_n^\circ(\mathbf{n} \cos 2\vartheta + \mathbf{a} \cos\vartheta) - NE_n^\circ[\mathbf{n} \times \mathbf{a}] \tag{1.14}$$

These are the values of the amplitudes of the reflected wave on the surface of the body that follow from the Fresnel formulae.

From the preceding formulae we can also derive relations for the total field. Denoting the total field on the surface of the body by $\mathbf{E}$ and $\mathbf{H}$ and their normal components by E_n and H_n and putting

$$\varkappa = \sqrt{\left(1 - \frac{\sin^2\vartheta}{\eta\mu}\right)} \tag{1.15}$$

we have

$$\sin^2\vartheta\,(\mathbf{E}-\mathbf{n}E_n) = \varkappa\sqrt{\left(\frac{\mu}{\eta}\right)}E_n\{\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})\}+H_n[\mathbf{n}\times\mathbf{a}] \tag{1.16}$$

$$\sin^2\vartheta\,[\mathbf{n}\times\mathbf{H}] = E_n\{\mathbf{a}-\mathbf{n}(\mathbf{a}\cdot\mathbf{n})\}+\varkappa\sqrt{\left(\frac{\eta}{\mu}\right)}H_n[\mathbf{n}\times\mathbf{a}] \tag{1.17}$$

If $|\eta\mu|\gg 1$, then $\varkappa=1$ approximately and the right-hand sides of (1.16) (1.17) are proportional to each other. In this case we have

$$\mathbf{E}-\mathbf{n}E_n = \sqrt{\left(\frac{\mu}{\eta}\right)}[\mathbf{n}\times\mathbf{H}] \tag{1.18}$$

The last relation does not involve the vector **a** and is thus independent of the direction of the incident wave. As shown by M. A. Leontovich, it holds not only in the illuminated region where the Fresnel formulae are applicable but also on the whole surface of the body.

From the formulae (1.16) and (1.17) the following relations can also be derived:

$$(\mathbf{a}\cdot\mathbf{E}) = \left(-\cos\vartheta+\varkappa\sqrt{\left(\frac{\mu}{\eta}\right)}\right)E_n \tag{1.19}$$

$$(\mathbf{a}\cdot\mathbf{H}) = \left(-\cos\vartheta+\varkappa\sqrt{\left(\frac{\eta}{\mu}\right)}\right)H_n \tag{1.20}$$

If the incident wave is plane so that the vector **a** has a fixed value, then the latter relations can be used instead of the Leontovich conditions (1.18). This is convenient when glancing ray incidence is considered and in this case the value $\sin\vartheta=1$ can be substituted in the expression (1.15) for $\varkappa$.

2. *Cross-section of a Beam of Reflected Rays*

In order to find the amplitude of the reflected wave at a certain distance from the body surface, it is necessary to have formulae for the cross-section of a beam reflected by an area $\mathrm{d}S$ of the surface and having travelled over a given path s after reflection. These formulae can be derived from well-known formulae of differential geometry.

Let the equations of the reflecting surface be:

$$x = x_0(u,v); \qquad y = y_0(u,v); \qquad z = z_0(u,v) \tag{2.01}$$

where u, v are the Gaussian coordinate parameters. The square of the element of arc on the surface can be written in the form

$$\mathrm{d}l^2 = g_{uu}\,\mathrm{d}u^2+2g_{uv}\,\mathrm{d}u\,\mathrm{d}v+g_{vv}\,\mathrm{d}v^2 = \sum_{u,\,v} g_{uv}\,\mathrm{d}u\,\mathrm{d}v \tag{2.02}$$

where the sum $\sum_{uv}$ is an abridged notation for the middle term of this equality.

We will use notations for the covariant and contravariant components of vectors and tensors, and raise and lower the indices by means of the metric tensor which enters into (2.02). We will write the surface element in the form

$$dS = \sqrt{g}\cdot du\, dv \tag{2.03}$$

Let us write the formulae for the components of the unit vector of the normal to the surface and for their derivatives with respect to u, v. We have

$$\sqrt{g}\cdot n_x = \frac{\partial y_0}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial y_0}{\partial v}\frac{\partial z_0}{\partial u} \qquad \text{etc.} \tag{2.04}$$

$$\frac{\partial n_x}{\partial u} = -\sum_v G_u^v \frac{\partial x_0}{\partial v} \qquad \text{etc.} \tag{2.05}$$

The last formula can be used as a definition of the quantity G_u^v, the mixed components of the second quadratic form of the surface. If R_1 and R_2 are the principal radii of curvature of the normal cross-section of the surface, then we have

$$K = \frac{1}{R_1 R_2} = G_u^u G_v^v - G_v^u G_u^v \tag{2.06}$$

$$\frac{1}{R_1} + \frac{1}{R_2} = -G = -G_u^u - G_v^v \tag{2.07}$$

The quantity K is the Gaussian curvature of the surface. We will also want a formula for the radius of curvature R_0 of the section of the surface in the plane of incidence of the ray. It can be shown that if $k\psi$ is the phase of the incident wave, and

$$(\text{grad}\,\psi)^2 = 1 \tag{2.08}$$

then

$$\sum_{u,v} g^{uv}\frac{\partial \psi_0}{\partial u}\frac{\partial \psi_0}{\partial v} = \sin^2\vartheta \tag{2.09}$$

where ϑ is the angle of incidence and the function differentiated is the value $\psi = \psi_0$ of the phase on the surface of the body. The quantity R_0 is then determined from the equation

$$\sum_{u,v} G^{uv}\frac{\partial \psi_0}{\partial u}\frac{\partial \psi_0}{\partial v} = -\frac{\sin^2\vartheta}{R_0} \tag{2.10}$$

Let us apply the above formulae to the calculation of the normal cross-section of a beam of rays reflected from the surface element $\mathrm{d}S$.

We consider the equations

$$\begin{aligned} x &= x_0 + s a_x^* \\ y &= y_0 + s a_y^* \\ z &= z_0 + s a_z^* \end{aligned} \tag{2.11}$$

in which s is a given quantity and x_0, y_0, z_0, a_x^*, a_y^*, a_z^* are functions of u, v determined from the equation (2.01) of the surface and from the relations

$$\mathbf{a}^* = \mathbf{a} - 2\mathbf{n}(\mathbf{a}\cdot\mathbf{n}) \tag{2.12}$$

where $\mathbf{n}$ is the vector of the normal at the point x_0, y_0, z_0. The quantity s is, evidently, the path the beam travelled after reflection. For constant s, equations (2.11) represent the equations of a surface parallel, in a certain sense, to the reflecting surface. If we vary u, v within the limits $(u, u+\mathrm{d}u)$, $(v, v+\mathrm{d}v)$ we obtain an element of the surface (2.11). This element can be considered as the section, by the "parallel" surface, of a beam of reflected rays arising from an element of the reflecting surface $\mathrm{d}S = \sqrt{g}\cdot\mathrm{d}u\,\mathrm{d}v$. In order to obtain a normal cross-section of the beam, we must project this section on a plane perpendicular to the reflected ray. Denoting the area of the normal section by $D(s)\,\mathrm{d}S$ we will have

$$D(s)\,\mathrm{d}S = \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v \tag{2.13}$$

from which:

$$D(s) = \frac{1}{\sqrt{g}} \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \tag{2.14}$$

We calculate this determinant under the assumption that the incident wave is plane and that, consequently the vector $\mathbf{a}$ is independent of u, v.

After rather complicated computations, which we omit here, the follow-

ing result is obtained:

$$D(s) = \cos\vartheta + 2s\left(-G + G\sum_{u,v} g^{uv}\frac{\partial\psi_0}{\partial u}\frac{\partial\psi_0}{\partial v} - \sum_{u,v} G^{uv}\frac{\partial\psi_0}{\partial u}\frac{\partial\psi_0}{\partial v}\right) + \\ + 4Ks^2\cos\vartheta \qquad (2.15)$$

Using expressions (2.06)–(2.10) given above, we can write:

$$D(s) = \cos\vartheta + 2s\left[\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\cos^2\vartheta + \frac{\sin^2\vartheta}{R_0}\right] + \frac{4s^2}{R_1R_2}\cos\vartheta \qquad (2.16)$$

where the values of R_0, R_1, R_2 are taken at the point where the reflection took place.

The quantity $D(s)/D(0)$ gives evidently the beam broadening, i.e., the ratio of the cross-section of the beam at the distances from the surface (measured along the ray) to its cross-section at the surface itself.

3. *The Electromagnetic Field of the Reflected Wave*

Let the field of the incident plane wave be equal to

$$\mathbf{E}^0 e^{i\varphi}, \qquad \mathbf{H}^0 e^{i\varphi} \qquad (3.01)$$

where $\mathbf{E}^0$ and $\mathbf{H}^0$ are constant amplitudes and

$$\varphi = k\psi = k(xa_x + ya_y + za_z) \qquad (3.02)$$

is the phase of the wave at a given point in space.

Introducing the value

$$\varphi_0 = k\psi_0 = k(x_0a_x + y_0a_y + z_0a_z) \qquad (3.03)$$

of the phase φ on the surface of the body we will have for the field of the incident wave on this surface the expressions

$$\mathbf{E}^0 e^{ik\psi_0}; \qquad \mathbf{H}^0 e^{ik\psi_0} \qquad (3.04)$$

The field of the reflected wave on the surface will be equal to

$$\mathbf{E}^* e^{ik\psi_0}; \qquad \mathbf{H}^* e^{ik\psi_0} \qquad (3.05)$$

where $\mathbf{E}^*$ and $\mathbf{H}^*$ are connected with $\mathbf{E}^0$ and $\mathbf{H}^0$ by the Fresnel formulae (1.13) and (1.14). As to the notations let us observe that in (1.13) and (1.14) we considered the phase factor $e^{ik\psi_0}$ to be included in $\mathbf{E}^0$, $\mathbf{H}^0$ and in $\mathbf{E}^*$, $\mathbf{H}^*$ but since this factor is the same on both sides of (1.13) and (1.14) it does not matter whether we mean by $\mathbf{E}^0$, $\mathbf{H}^0$ and $\mathbf{E}^*$, $\mathbf{H}^*$ in these equations the total expressions (3.04) and (3.05) or their amplitudes.

In the notation of this section, $\mathbf{E}^0$ and $\mathbf{H}^0$ are constants and $\mathbf{E}^*$ and $\mathbf{H}^*$ are slowly varying functions of the coordinates of a point on the surface.

Let F denote one of the components of the reflected wave field. The value of F on the surface will be

$$F = f(u, v)\, e^{ik\psi_0(u, v)} \tag{3.06}$$

where $f(u, v)$ is a slowly varying function and k is a large parameter. In order to find F at a certain distance s from the surface, we must know the solution of the wave equation

$$\Delta F + k^2 F = 0 \tag{3.07}$$

which satisfies the radiation condition and the boundary condition (3.06) on the surface. Considering k a large parameter, it is possible to write down an approximate solution in an explicit form.

Indeed, let us consider the expression:

$$F = f(u, v) \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik(\psi_0 + s)} \tag{3.08}$$

The quantities u, v, s can be interpreted as curvilinear coordinates of a space point which are connected with the rectangular coordinates x, y, z by the relation (2.11). The geometrical meaning of these curvilinear coordinates is evident: the parameters u, v determine the position of that surface point from which the ray, reaching after reflection the point x, y, z, is reflected; the quantity s is the distance the ray has travelled after reflection.

Thus the quantity F in (3.08) can be interpreted as a function of the space point. It is obvious that this function takes the value (3.06) on the surface $s = 0$. It is also evident that it satisfies the radiation condition and corresponds to a scattered wave. Moreover, if the parameter k is large, then F approximately satisfies the wave equation. Indeed, it can be shown that from the definitions of ψ_0 and $D(s)$ and from the expressions (2.11) for the rectangular coordinates follow the equalities

$$\{\text{grad}\,(\psi_0 + s)\}^2 = 1 \tag{3.09}$$

$$\text{div}\left\{f^2 \frac{D(0)}{D(s)} \text{grad}\,(\psi_0 + s)\right\} = 0 \tag{3.10}$$

Using these equations it is easy to verify that if F is inserted into (3.07) the second and first order terms in k drop out and only zero order terms remain.

Independently of the reasoning just exposed, the validity of (3.08) follows from geometrical optics considerations. In fact, this expression must give the reflected wave. Now the phase of the reflected wave is obviously equal to $k(\psi_0 + s)$. To determine the amplitude we must take into account that if we move along a thin bundle of reflected rays, the amplitude must vary

in inverse proportion to the square root of the beam cross-section, as given by (3.08).

Thus this formula gives the field of the reflected wave at a distance s from the surface when the field on the surface itself is known.

Applying this formula to the components of the electric and magnetic field we obtain the expressions

$$\mathbf{E} = \mathbf{E}^*(u, v) \cdot \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik(\nu_0+s)} \tag{3.11}$$

$$\mathbf{H} = \mathbf{H}^*(u, v) \cdot \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik(\nu_0+s)} \tag{3.12}$$

where $\mathbf{E}^*(u, v)$ and $\mathbf{H}^*(u, v)$ are the surface values of the field amplitudes obtained from the Fresnel formulae.

The formulae we derived for the field are natural combinations of the laws of reflection and the laws of geometrical (ray) optics. Both, separately, were known over a hundred years ago. Fresnel found his reflection laws about 1820 and Hamilton found the ray optic laws about 1830. In particular, Hamilton knew that the quantity, corresponding to our $D(s)$ is a second degree polynomial in s. However, we have not been able to find in the literature any indication concerning the application of these results to the approximate representation of a reflected electromagnetic wave.

4. *The Diffraction Laws in the Penumbra Region*†

In the introduction, we already mentioned that the incident and the reflected wave become inseparable from each other near the geometrical boundary of the shadow, in the region of oblique ray incidence, and the Fresnel formulae become inapplicable. On the basis of our work, (Chapter 5) we shall now consider the derivation of diffraction formulae which give the field in this region as well as in the penumbra and in the shadow regions.

Let us imagine a convex body on which a plane wave falls in the direction of the x-axis. Let us choose a point on the surface of the body which lies on the geometrical boundary of the shadow and let us take it as the origin of the coordinates. We direct the z-axis along the normal to the surface (towards the air). Since on the boundary of the shadow the normal is perpendicular to the direction of the wave, our x-and z-axes will be perpendicular to each other. We direct the y-axis so as to obtain a right-handed coordinate system.

† Section 4 is a summary of our paper [5].

In the neighbourhood of the given point the equation of the surface will be of the form

$$z+\frac{1}{2}(ax^2+2bxy+cy^2) = 0 \tag{4.01}$$

in which

$$a \geqslant 0; \quad c \geqslant 0 \quad ac-b^2 \geqslant 0 \tag{4.02}$$

The radius of curvature of the section of the surface will be equal to

$$R_0 = \frac{1}{a} \tag{4.03}$$

In the following we will introduce a large parameter m according to the formula:

$$m = \sqrt[3]{\left(\frac{kR_0}{2}\right)} = \sqrt[3]{\left(\frac{k}{2a}\right)} \tag{4.04}$$

and we will solve our problem neglecting quantities of order $\frac{1}{m^2}$ in comparison with unity.

Our aim is to find the electromagnetic field at distances from the origin that are small as compared to the radius of curvature R_0.

With these assumptions, each field component will be of the form

$$F = \mathrm{e}^{ikx} F^* \tag{4.05}$$

where F^* satisfies the differential equation

$$\frac{\partial^2 F^*}{\partial z^2}+2ik\frac{\partial F^*}{\partial x} = 0 \tag{4.06}$$

All the field components can be expressed in terms of H_y and H_z according to the formulae

$$\left.\begin{aligned} E_x &= \frac{i}{k}\left(\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z}\right) \\ E_y &= H_z \\ E_z &= -H_y \\ H_x &= \frac{i}{k}\left(\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z}\right) \end{aligned}\right\} \tag{4.07}$$

which can be considered as simplified Maxwell equations.

The approximate boundary conditions for the field in air near a good-conducting body have been established by Leontovich [21]. They are valid under the assumptions:

$$|\eta\mu| \gg 1; \qquad kR_0 \left|\sqrt{(\eta\mu)}\right| \gg 1 \tag{4.08}$$

and have the form (see (1.18))

$$\mathbf{E} - \mathbf{n}E_n = \sqrt{\left(\frac{\mu}{\eta}\right)}[\mathbf{n}\times\mathbf{H}] \tag{4.09}$$

In the following, we will suppose $\mu=1$. The vector components of the normal in (4.09) are determined from equation (4.01) for the surface. We can write, approximately

$$n_x = ax+by; \qquad n_y = bx+cy; \qquad n_z = 1 \tag{4.10}$$

the squares of n_x and n_y being negligible in comparison with unity. We will consider the quantities n_x, n_y, $\dfrac{1}{m}$, $\dfrac{1}{\sqrt{\eta}}$ to be of the same order of smallness.

Under these assumptions, we can derive from (4.07) and (4.09) boundary conditions which contain only H_y and H_z. They will be of the form

$$H_z = -n_y H_y \tag{4.11}$$

$$\frac{\partial H_y}{\partial z} + ik\left(n_x + \frac{1}{\sqrt{\eta}}\right)H_y = \frac{\partial H_z}{\partial y} \tag{4.12}$$

Owing to the smallness of the quantity n_y, the right-hand sides of these equations are correction terms. In the first approximation, they can be replaced by zero and the following more simple boundary conditions can be used

$$H_z = 0 \tag{4.13}$$

$$\frac{\partial H_y}{\partial z} + ik\left(n_x + \frac{1}{\sqrt{\eta}}\right)H_y = 0 \tag{4.14}$$

In the second approximation we can insert into the right-hand sides of (4.11) and (4.12) the values H_y and H_z obtained by solving the differential equations with the boundary conditions (4.13) and (4.14).

In addition to the above conditions the solution must also satisfy conditions at infinity. These require that the part of the solution which corresponds to the plane wave should have a given amplitude at infinity.

The mathematical problem formulated has a unique solution which we

will give here omitting all the calculations and confining ourselves to definitions.

Apart from the factor e^{ikx} the field will depend on the coordinates only through the quantities:†

$$\xi = m(ax+by) \tag{4.15}$$

$$\zeta = 2am^2\left[z+\frac{1}{2}(ax^2+2bxy+cz^2)\right] \tag{4.16}$$

of which the second vanishes on the surface. The constants, characterizing the electric properties of the reflecting surface, enter into the expressions for the field through the quantity‡

$$q = \frac{im}{\sqrt{\eta}}; \qquad m = \sqrt[3]{\left(\frac{k}{2a}\right)} \tag{4.17}$$

Ultimately the field is expressed in terms of one universal function $V_1(\xi, \zeta, q)$ independent of the shape of the surface, and its limiting value:

$$V_2(\xi, \zeta) = V_1(\xi, \zeta, \infty) \tag{4.18}$$

The function V_1 can be represented as a definite integral containing the complex Airy functions $w_1(t)$ and $w_2(t)$. These latter are defined as the solutions of the differential equation

$$w''(t) = tw(t) \tag{4.19}$$

which have, for large negative t, the asymptotic expressions

$$w_1(t) = \frac{1}{\sqrt[4]{(-t)}}\exp\left(i\frac{2}{3}(-t)^{3/2}+i\frac{\pi}{4}\right) \tag{4.20}$$

$$w_2(t) = \frac{1}{\sqrt[4]{(-t)}}\exp\left(-i\frac{2}{3}(-t)^{3/2}-i\frac{\pi}{4}\right) \tag{4.21}$$

The expression for V_1 has the form

$$V_1(\xi, \zeta, q) = \frac{i}{2\sqrt{\pi}}\int_c e^{i\xi t}\left[w_2(t-\zeta)-\frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\,w_1(t-\zeta)\right]dt \tag{4.22}$$

† The correction terms will also contain linearly the coordinate y.

‡ If we used relations (1.19) and 1.20) instead of the Leontovich conditions (1.18) we would obtain for q the somewhat more exact expression $q = \frac{im}{\eta}\sqrt{(\eta-1)}$.

where the contour C goes along the ray arc $t = \frac{2}{3}\pi$ from infinity to zero and along the ray arc $t = -\frac{1}{3}\pi$ from zero to infinity.†

For $\zeta=0$ (on the surface of the body) the expression for V_1 simplifies and takes the form

$$V_1(\xi, 0, q) = \frac{1}{\sqrt{\pi}} \int_C e^{i\xi t} \frac{dt}{w'(t) - q w(t)} \tag{4.23}$$

This function is tabulated for a number of values of q; the tables for $q=0$ (absolutely-conducting body) are published in Chapter 2.

Having the definition of the function $V_1(\xi, \zeta, q)$, we can write the expression for the field. In order to do this, we introduce the functions

$$\Psi = e^{-i\varphi} V_1(\xi, \zeta, q) \tag{4.24}$$

$$\Phi = e^{-i\varphi} V_2(\xi, \zeta) \tag{4.25}$$

where

$$\varphi = \xi\zeta - \frac{1}{3}\xi^3 \tag{4.26}$$

With their aid we form the following expressions:

$$P = -i\frac{b}{a}\frac{\partial \Psi}{\partial \xi} + \left(i\frac{b}{a}q + \frac{ac-b^2}{a}my\right)(\Phi - \Psi) \tag{4.27}$$

$$Q = i\frac{b}{a}\frac{\partial \Phi}{\partial \zeta} + \left(-i\frac{b}{a}q + \frac{ac-b^2}{a}my\right)(\Phi - \Psi) \tag{4.28}$$

Then the components H_y and H_z of the magnetic field will be equal to

$$H_y = H_y^{\circ} e^{ikx}\Psi + \frac{1}{m} H_z^{\circ} e^{ikx} Q \tag{4.29}$$

$$H_z = \frac{1}{m} H_y^{\circ} e^{ikx} P + H_z^{\circ} e^{ikx}\Phi \tag{4.30}$$

where H_y° and H_z° are the amplitudes of the incident wave. Each of the functions Φ, Ψ, P, Q satisfies a differential equation of the form (4.06) and all four functions will be of the same order of magnitude.‡ Since m is a

† Tables of the function (4.22) for $q=0$ and $q=\infty$ have since been published by M. Belkina and P. Aszilant [27].

‡ The quantities Φ, Ψ, P, Q are the same as Φ^*, Ψ^*, P^*, Q^* in Chapter 5.

large parameter, the terms in Φ and Ψ will be the principal terms, and the terms in P and Q will be small corrections. The field components E_x and H_x will be of the same order as the correction terms, namely

$$E_x = -\frac{i}{m} H_y^\circ e^{ikx} \frac{\partial \Psi}{\partial \zeta} \tag{4.31}$$

$$H_x = \frac{i}{m} H_z^\circ e^{ikx} \frac{\partial \Phi}{\partial \zeta} \tag{4.32}$$

As for the remaining components of the electric field, they will be equal to

$$E_y = H_z; \qquad E_z = -H_y \tag{4.33}$$

by virtue of the simplified Maxwell equations (4.07).

We have thus determined all the field components.

5. *Investigation of the Expressions for the Field in the Shadow and in the Illuminated Regions*

The diffraction formulae we derived above give the field near a certain point on the surface of a conducting body on the geometrical boundary of the shadow. We will show that they give a continuous transition from the field corresponding to the Fresnel formulae (for the illuminated region) to total shadow. Let us begin with the shadow region.

The integral (4.22) can be represented as the sum of residues at the poles corresponding to the roots of the denominator of the integrand. We have

$$V_1(\xi, \zeta, q) = i\, 2\sqrt{\pi} \sum_{s=1}^{\infty} e^{i\xi t_s} \frac{w_1(t_s - \zeta)}{w_1^2(t_s)\,(t_s - q^2)} \tag{5.01}$$

where t_s is a root of the equation:

$$w_1'(t_s) - q w_1(t_s) = 0 \tag{5.02}$$

The roots t_s lie near the ray arc $t = \dfrac{\pi}{3}$ and increase in absolute value. For sufficiently large positive values of $\xi - \sqrt{\zeta}$, we can neglect in the series (5.01) all the terms except the first. If, moreover, the asymptotic expression (4.20) for w_1 is used and if ζ is considered to be large as compared to t_1, then the following approximate expression for V_1 is obtained:

$$V_1(\xi, \zeta, q) = \frac{e^{i\frac{3\pi}{4}}\, 2\sqrt{\pi}}{w_1^2(t_1)\,(t_1 - q^2)} \cdot e^{i\frac{2}{3}\zeta^{3/2}} \cdot e^{i(\xi - \sqrt{\zeta})t_1} \tag{5.03}$$

The quantity t_1 has for $q=0$ and $q=\infty$ the following values:

$$t_1 = 1{\cdot}01879{\cdot}e^{i\frac{\pi}{3}} \qquad (q = 0) \tag{5.04}$$

$$t_1 = 2{\cdot}33811{\cdot}e^{i\frac{\pi}{3}} \qquad (q = \infty) \tag{5.05}$$

In all cases both the real and the imaginary parts of t_1 are positive. Hence it follows that when $(\xi-\sqrt{\zeta})$ increases the functions V_1 and V_2 and the quantities Φ, Ψ, P and Q connected with them and, consequently, the field, will decrease exponentially.

Let us observe that the equation $\xi-\sqrt{\zeta}=0$ yields the geometrical boundary of the shadow. Increasing positive values of the quantity $\xi-\sqrt{\zeta}$ correspond to points lying farther and farther in the shadow region.

In the region where the quantity $\xi-\sqrt{\zeta}$ is not large (it can be of either sign) we have the penumbra region. We will not dwell on methods of computing the function V_1 in this region; we will only observe that in this region the function V_1 and, therefore, the field vary continuously.

Now, let us consider the illuminated region where the quantity $\xi-\sqrt{\zeta}$ is large and negative. In this case, it is practically impossible to use the series (5.01) for V_1 and it is necessary to return to the integral (4.22). The term containing the function $w_2(t-\zeta)$ in this integral can be evaluated exactly. It yields

$$\frac{i}{2\sqrt{\pi}}\int_c e^{i\xi t}\, w_2(t-\zeta)\, dt = e^{i\varphi} \tag{5.06}$$

where φ has the value:

$$\varphi = \xi\zeta-\frac{1}{3}\xi^3 \tag{5.07}$$

which coincides with (4.26). According to (4.24) and (4.25) this term in V_1 and V_2 gives the term unity in the Φ and Ψ functions; in the field expressions this term corresponds therefore to an incident wave.

The second term can be evaluated according to the stationary phase method as shown in Chapter 5.. The phase extremum is obtained for $\sqrt{(-t)}=p$ where

$$p = \frac{1}{3}\left(\sqrt{(\xi^2+3\zeta)}-2\xi\right) \tag{5.08}$$

It is convenient to introduce the special notation:

$$\sigma = \sqrt{(\xi^2+3\zeta)} \tag{5.09}$$

for the square root in the above formula. Let us observe that the quantity p has the same sign as $\sqrt{\zeta}-\xi$ so that $p>0$ corresponds to the illuminated region, $p=0$ to the geometrical boundary of the shadow and $p<0$ to the shadow. We are interested now in large positive values of p. In this case the use of the stationary phase method gives for the total quantity V_1 the expression

$$V_1(\xi, \zeta, q) = e^{i\varphi} - e^{i\varphi^*} \cdot \sqrt{\left(\frac{p}{\sigma}\right)} \cdot \frac{q-ip}{q+ip} \tag{5.10}$$

where the phase φ is equal to (5.07) and the phase φ^* equals

$$\varphi^* = \frac{1}{27}(4\sigma^3 - 3\sigma^2\xi - 2\xi^3) \tag{5.11}$$

Let us note that the phase difference $\varphi^* - \varphi$ is equal to

$$\varphi^* - \varphi = \frac{4}{27}(\sigma+\xi)(\sigma-2\xi)^2 = 2(\sigma-p)p^2 \tag{5.12}$$

For $\zeta=0$ we have $\sigma=p=-\xi$, so that $\varphi^*-\varphi$ vanishes on the surface of the body.

The quantity V_2 is obtained from (5.10) for $q=\infty$. The functions Φ and Ψ connected with V_1 and V_2 will be approximately equal to

$$\Psi = 1 - e^{i(\varphi^*-\varphi)} \sqrt{\left(\frac{p}{\sigma}\right)} \cdot \frac{q-ip}{q+ip} \tag{5.13}$$

$$\Phi = 1 - e^{i(\varphi^*-\varphi)} \sqrt{\left(\frac{p}{\sigma}\right)} \tag{5.14}$$

The field expressions involve not only the functions Ψ and Φ themselves but also their derivatives with respect to ζ. In forming the derivatives all the factors, except the phase factor, can be considered as constant. Since

$$\frac{\partial(\varphi^*-\varphi)}{\partial\zeta} = \frac{2}{3}\sigma - \frac{4}{3}\xi = 2p \tag{5.15}$$

we will have

$$\frac{\partial\Psi}{\partial\zeta} = 2ip(\Psi-1); \qquad \frac{\partial\Phi}{\partial\zeta} = 2ip(\Phi-1) \tag{5.16}$$

Calculating the quantities P and Q by means of these values we obtain

$$P = Q = \frac{2ip}{q+ip}\left(\frac{b}{a}p - \frac{ac-b^2}{a}my\right)\sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i(\varphi^*-\varphi)} \tag{5.17}$$

To obtain the field we have only to insert the expressions found into the formulae (4.29)–(4.32). It is convenient to denote the phase of the reflected wave by a single letter

$$\chi = kx + \varphi^* - \varphi \tag{5.18}$$

With this notation we have

$$H_y = H_y^\circ e^{ikx} - H_y^\circ \frac{q-ip}{q+ip} \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} + \\ + \frac{1}{m} H_z^\circ \frac{2ip}{q+ip} \left(\frac{b}{a} p - \frac{ac-b^2}{a} my\right) \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{5.19}$$

$$H_z = H_z^\circ e^{ikx} - H_z^\circ \sqrt{\left(\frac{p}{\sigma}\right)} e^{i\chi} + \\ + \frac{1}{m} H_y^\circ \frac{2ip}{q+ip} \left(\frac{b}{a} p - \frac{ac-b^2}{a} my\right) \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{5.20}$$

$$E_x = -\frac{1}{m} H_y^\circ \cdot 2p \frac{q-ip}{q+ip} \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{5.21}$$

$$H_x = \frac{1}{m} H_z^\circ \cdot 2p \sqrt{\left(\frac{p}{\sigma}\right)} e^{i\chi} \tag{5.22}$$

and, moreover, $E_y = H_z$; $E_z = -H_y$.

The first term in (5.19) and also in (5.20) obviously gives the incident wave and the remaining terms yield the reflected wave. In the next section we will show that the reflected wave exactly corresponds to the Fresnel formula with a correction for beam broadening.

6. *Comparison of the Diffraction and the Fresnel Formulae for the Illuminated Region*

Let us now turn to the Fresnel formulae. Putting $\mu=1$ in the Fresnel coefficients and considering $\sqrt{\eta}$ as a large quantity and $\cos\vartheta$ as a small quantity (of the order of $1/\sqrt{\eta}$), we obtain for N and M the expressions

$$N = \frac{\sqrt{\eta}\cdot\cos\vartheta - 1}{\sqrt{\eta}\cdot\cos\vartheta + 1}; \qquad M = -1 \tag{6.01}$$

In the Fresnel formulae (1.13) and (1.14) we must put $a_x=1, a_y=a_z=0$ and consider n_x and n_y as small quantities, the squares of which can be neglected. Then these formulas give for the electric field

$$\left.\begin{aligned} E_x^* &= -2Nn_x H_y^\circ \\ E_y^* &= -H_z^\circ - (N+1)n_y H_y^\circ \\ E_z^* &= -NH_y^\circ + (N+1)n_y H_z^\circ \end{aligned}\right\} \tag{6.02}$$

and for the magnetic field:

$$\left.\begin{aligned} H_x^* &= -2n_x H_z^\circ \\ H_y^* &= NH_y^\circ-(N+1)n_y H_z^\circ \\ H_z^* &= -H_z^\circ-(N+1)n_y H_y^\circ \end{aligned}\right\} \tag{6.03}$$

In order to obtain the field of the reflected wave at a certain distance from the surface, it is necessary, according to (3.11) and (3.12), to multiply these expressions by the factor

$$\sqrt{\left(\frac{D(0)}{D(s)}\right)}\cdot e^{ik(x_0+s)} \tag{6.04}$$

The values of all the variables, except s, are to be taken at that point x_0, y_0, z_0 where the reflection of the ray going after reflection through the point x, y, z takes place. Since the equation of the reflecting surface is

$$z_0+\frac{1}{2}(ax_0^2+2bx_0y_0+cy_0^2) = 0 \tag{6.05}$$

we have

$$n_x = ax_0+by_0; \qquad n_y = bx_0+cy_0; \qquad n_z = 1 \tag{6.06}$$

When evaluating $D(s)$ according to the general formula (2.23), we must neglect the last term since we are interested in the field at distances which are small as compared with the radius of curvature. The remaining terms yield

$$D(s) = \cos\vartheta+2as = 2as-ax_0-by_0 \tag{6.07}$$

In order to make a comparison between the diffraction formulae (5.19)–(5.22) and the Fresnel formulae (6.02), (6.03), we must establish the connection between the quantities x_0, y_0, s and the coordinates x, y, z (or the quantities ξ, ζ, y). This connection is given by the formulae (2.11), which in our case take the form

$$\begin{aligned} x &= x_0+s-2sn_x^2 \\ y &= y_0-2sn_xn_y \\ z &= z_0-2sn_xn_z \end{aligned} \tag{6.08}$$

Solving these equations, approximately, with respect to x_0, y_0, s we obtain

$$\left.\begin{aligned} ax_0+by_0 &= \frac{2\xi-\sigma}{3m} = -\frac{p}{m} \\ y_0 &= y \\ s &= \frac{\sigma+\xi}{3am} = \frac{\sigma-p}{2am} \end{aligned}\right\} \tag{6.09}$$

Hence

$$n_x = -\frac{p}{m}; \qquad n_y = -\frac{b}{a}\frac{p}{m}+\frac{ac-b^2}{a}y \tag{6.10}$$

Further, according to (5.12), (5.13), the phase χ is equal to

$$\chi = kx+\varphi^*-\varphi = kx+2(\sigma-p)p^2 = k(x+2sn_x^2) = k(x_0+s) \tag{6.11}$$

that is, it equals the phase of the reflected wave calculated from geometrical optics. Let us now calculate the quantity $D(s)$. Inserting the expressions (6.09) into (6.07), we obtain

$$D(s) = \frac{\sigma}{m} \tag{6.12}$$

and of course

$$D(0) = \cos\vartheta = \frac{p}{m} \tag{6.13}$$

The last three formulae give

$$\sqrt{\left(\frac{p}{\sigma}\right)}\cdot e^{i\chi} = \sqrt{\left(\frac{D(0)}{D(s)}\right)}\cdot e^{ik(x_0+s)} \tag{6.14}$$

Thus, the factor (6.14) involved in all the expressions for the reflected wave in the diffraction formulae (5.19)–(5.22) coincides with the factor entering into the formulae (3.08)–(3.09) which represent generalizations of the Fresnel formulae. The quantity

$$\sqrt{\left(\frac{p}{\sigma}\right)} = \sqrt{\left(\frac{1}{3}-\frac{2\xi}{3\sigma}\right)} \tag{6.15}$$

in (6.14) gives the correction for beam broadening.

We have still to verify that all the other quantities in (5.19)–(5.22) agree with the Fresnel quantities.

According to (4.17) and (6.13), we have

$$q = \frac{im}{\sqrt{\eta}}; \qquad p = m\cos\vartheta \tag{6.16}$$

Therefore

$$\frac{q-ip}{q+ip} = \frac{1-\sqrt{\eta}\cdot\cos\vartheta}{1+\sqrt{\eta}\cdot\cos\vartheta} = -N \tag{6.17}$$

where N is the Fresnel coefficient (6.01).*

* The value $q = \frac{im}{\eta}\sqrt{(\eta-1)}$ leads to a rather more exact value of N, namely

$$N = \frac{\eta\cos\vartheta-\sqrt{(\eta-1)}}{\eta\cos\vartheta+\sqrt{(\eta-1)}}$$

Defining n_x and n_y by the formula (6.10) and N by the expression (6.17) we can write our expressions (5.19)–(5.22) for the field as follows

$$H_y = H_y^\circ e^{ikx} + [NH_y^\circ - (N+1)n_y H_z^\circ] \cdot \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{6.18}$$

$$H_z = H_z^\circ e^{ikx} + [-H_z^\circ - (N+1)n_y H_y^\circ] \cdot \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{6.19}$$

$$E_x = -2Nn_x H_y^\circ \sqrt{\left(\frac{p}{\sigma}\right)} \cdot e^{i\chi} \tag{6.20}$$

$$H_x = -2n_x H_z^\circ \sqrt{\left(\frac{p}{\sigma}\right)} e^{i\chi} \tag{6.21}$$

Comparing these expressions with the Fresnel formulae (6.02) and (6.03) we see that the factors of (6.14) coincide exactly with their Fresnel values H_y^*, H_z^*, E_x^*, H_x^*. The equalities $E_y = H_z$ and $E_z = -H_y$ are satisfied both in the case of our formulae and in the case of the Fresnel formulae.

We have thus shown that in that part of the illuminated region where the angle made by the incident ray with the surface of the reflecting body is small our formulae go over to the Fresnel formulae generalized by the introduction of the broadening factor (6.14).

Moreover, our formulae give the diffraction picture in the penumbra and the shadow regions.

CHAPTER 7

FRESNEL DIFFRACTION FROM CONVEX BODIES†

THE method of approximate solution of diffraction problems based on Huygens principle permits us, as is well known, to determine the field of a wave which is diffracted by a thin opaque screen. This field is expressed by Fresnel's integrals.

However, in the case when the diffracting body has a finite curvature (radius of curvature large with respect to the wavelength) the problem of an approximate determination of the field in the region of the geometrical boundary of the shadow at large distances from the body remained unsolved; in particular it remained uncertain whether in this case the same expressions for the field (the Fresnel integrals), which may be constructed in analogy with the case of an infinitely thin screen, are still applicable. In the present paper we show by considering the example of diffraction from a sphere, that for a body with finite curvature the main term in the expression for the field behind this body is given by the Fresnel integrals. This term does not depend on the material of the diffracting body (as is also the case in the usual Fresnel diffraction). To the main term there is added, however, an additional term, which constitutes a sort of background, on which the Fresnel fringes are superposed. This additional term (and consequently the background) depends, unlike the main term, on the electrical properties of the body round which the wave is diffracted.

1. *Formulae for the Attenuation Factor*

We take as point of departure the diffraction formulae, which were developed in one of our papers on radiowave propagation (Chapter 12). We have to summarize here the main results of this paper.

The field of a point source (dipole), situated at some distance from the surface of the sphere, is expressible in terms of two functions U and W,

† Fock, 1948.

which represent solutions of the wave equation

$$\Delta U + k^2 U = 0 \tag{1.01}$$

and have at the source-point a singularity of the form

$$U = \frac{e^{ikR}}{R} + U^0 \tag{1.02}$$

where U^0 remains finite for $kR \to 0$ (R is the distance from the source).

The equations defining U, W differ in the form of the boundary conditions, which we are not going to discuss here.

Let r, ϑ, φ be spherical coordinates with origin at the centre of the sphere, the dipole being situated on the poar axis. The quantity $s = a\vartheta$, where a is the radius of the sphere, gives the distance from the source to the observation point, measured along a great circle of the sphere. The height of the source above the sphere is denoted by h_1, the height of the observation point by h_2. Further we introduce the parameter

$$m = \sqrt[3]{\left(\frac{ka}{2}\right)} \tag{1.03}$$

which we assume to be large, and put

$$x = \sqrt[3]{\left(\frac{k}{2a^2}\right)} s = m \frac{s}{a} = m\vartheta \tag{1.04}$$

$$y_1 = \frac{kh_1}{m}; \qquad y_2 = \frac{kh_2}{m}. \tag{1.05}$$

The complex dielectric constant of the material of the sphere we denote by η, and assume that $|\eta| \gg 1$. Finally we put

$$q = \frac{im}{\sqrt{(\eta+1)}}; \qquad q_1 = im\sqrt{(\eta-1)} \tag{1.06}$$

In our paper we have shown, that in the vicinity of the surface of the sphere (that is, at distances small compared to its radius), the functions U and W can be expressed according to the formulae

$$U = \frac{e^{i}}{\sqrt{(sa \sin s/a)}} \cdot V(x, y_1, y_2, q) \tag{1.07}$$

$$W = \frac{e^{iks}}{\sqrt{(sa \sin s/a)}} \cdot V(x, y_1, y_2, q_1) \tag{1.08}$$

by an attenuation factor V which can be represented, for $y_1 < y_2$, in the form of a contour integral

$$V(x, y_1, y_2, q) = e^{-i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_C e^{ixt} F(t, y_1, y_2, q)\, dt \tag{1.09}$$

The function F in the integrand may be written in the form

$$F = w_1(t-y_2)\left[v(t-y_1) - \frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}\, w_1(t-y_1)\right] \tag{1.10}$$

or in the form

$$F = \frac{i}{2}\, w_1(t-y_2)\left[w_2(t-y_1) - \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\, w_1(t-y_1)\right] \tag{1.11}$$

Here $w_1(t)$ and $w_2(t)$ are the complex Airy functions, which represent solutions of the differential equation

$$w''(t) = tw(t) \tag{1.12}$$

having for large negative t the following asymptotic expressions

$$\left.\begin{aligned} w_1(t) &= e^{i\frac{\pi}{4}} (-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} \\ w_2(t) &= e^{-i\frac{\pi}{4}} (-t)^{-\frac{1}{4}} e^{-i\frac{2}{3}(-t)^{3/2}} \end{aligned}\right\} \tag{1.13}$$

Formulae (1.10) involves also one of the functions $u(t)$, $v(t)$, which are defined by the equations

$$w_1(t) = u(t) + iv(t); \qquad w_2(t) = u(t) - iv(t) \tag{1.14}$$

For t real, both functions $u(t)$, $v(t)$ are real. For all values of t we have

$$w_1\left(t\, e^{i\frac{2}{3}\pi}\right) = e^{i\frac{\pi}{3}}\, w_2(t); \qquad w_1\left(t\, e^{i\frac{4}{3}\pi}\right) = 2\, e^{i\frac{\pi}{6}}\, v(t) \tag{1.15}$$

The contour C in the integral (1.09) encloses in the positive direction the first quadrant of the complex t-plane (in this quadrant all the poles of the integrand are located). We can choose, for instance, as the contour C a broken line, which goes from $\infty e^{i\frac{2}{3}\pi}$ to 0 and from 0 to ∞.

2. *Transformation of the Attenuation Factor*

In Chapters 5 and 12 we investigate the attenuation factor V, firstly in the illuminated region, where the reflection formula corresponding to geometrical optics holds, secondly in the region of the shadow, where

the amplitudes of the field decay exponentially, and finally in the transition region near the surface of the sphere (region of half-shadow or penumbra region). The region of the shadow cone is not investigated there, and the aim of the present chapter is to derive approximations for this region.

The shadow cone is the cone whose generators are tangential to the sphere, and whose apex is at the source point. The equation of the shadow cone may be written in the form

$$\sqrt{(b^2-a^2)}+\sqrt{(r^2-a^2)} = \sqrt{(r^2+b^2-2rb\cos\vartheta)} \tag{2.01}$$

or, introducing the variables x, y_1, y_2, and neglecting small quantities

$$\sqrt{y_1}+\sqrt{y_2} = x \tag{2.02}$$

Thus, we have to investigate the attenuation factor V for the case when the quantities x, y_1, y_2 are very large, but the difference

$$\xi = x-\sqrt{y_1}-\sqrt{y_2} \tag{2.03}$$

remains finite. We note that positive values of ξ correspond to the shadow region and negative values of ξ to the illuminated region.

For the quantity F in the integral (1.09) for V we may take one of the two expressions (1.10) or (1.11), which are identically equal. Let us divide the contour C of the integral (1.09) into two segments, one of which from $\infty\, e^{i\frac{2\pi}{3}}$ to 0 we denote by C_1, and the other, that from 0 to ∞, by C_2. On the first segment we use for F the expression (1.11), on the second segment the expression (1.10). Then we may write

$$V = \Phi+\Psi \tag{2.04}$$

where

$$\Phi = \sqrt{\left(\frac{x}{\pi}\right)}\cdot e^{-i\frac{\pi}{4}}\left[\frac{i}{2}\int_{C_1} e^{ixt}\, w_1(t-y_2)w_2(t-y_1)\, dt + \right.$$
$$\left. +\int_{C_2} e^{ixt}\, w_1(t-y_2)v(t-y_1)\, dt\right] \tag{2.05}$$

$$\Psi = -\sqrt{\left(\frac{x}{\pi}\right)}\, e^{-i\frac{\pi}{4}}\left[\frac{i}{2}\int_{C_1} e^{ixt}\frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\, w_1(t-y_1)w_1(t-y_2)\, dt + \right.$$
$$\left. +\int_{C_2} e^{ixt}\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}\, w_1(t-y_1)w_1(t-y_2)\, dt\right] \tag{2.06}$$

The integrals in the expression for Φ do not depend on the parameter q, which appears only in Ψ. Consequently, Φ does not depend on the electrical

properties of the diffracting body; they influence only the quantity Ψ. We shall see that Φ corresponds to the Fresnel part of the diffraction and Ψ to the background, on which the Fresnel diffraction pattern is superposed.

3. *Evaluation of the Integral Φ*

In the expression (2.05) for Φ we may replace the integration over C_1 by an integration from $-\infty$ to 0. Using the relation $w_2 = w_1 - 2iv$ we obtain

$$\Phi = \Phi_1 + \Phi_2 \tag{3.01}$$

where

$$\Phi_1 = \frac{1}{2}\sqrt{\left(\frac{x}{\pi}\right)} \cdot e^{i\frac{\pi}{4}} \int_{-\infty}^{0} e^{ixt}\, w_1(t-y_2) w_1(t-y_1)\, dt \tag{3.02}$$

$$\Phi_2 = \sqrt{\left(\frac{x}{\pi}\right)} \cdot e^{-i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{ixt}\, w_1(t-y_2) v(t-y_1)\, dt \tag{3.03}$$

Let us calculate the integral Φ_2. For this purpose we make use of the following integral representation for $w_1(t-y_2)$:

$$w_1(t-y_2) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{(t-y_2)z - \frac{1}{3}z^3}\, dz \tag{3.04}$$

where the contour Γ consists of two segments from $-i\infty$ to 0 and from 0 to ∞. We observe that on the contour Γ we have: $Re(z) \geqslant 0$. After the substitution of (3.04) into (3.03), we can perform the integration over t with help of the formula

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{(z+ix)t}\, v(t-y_1)\, dt = \exp\left[y_1(z+ix) + \frac{1}{3}(z+ix)^3\right] \tag{3.05}$$

which holds for $Re(z) \geqslant 0$. We thus obtain

$$\Phi_2 = \sqrt{\left(\frac{x}{\pi}\right)} e^{-i\frac{\pi}{4}} e^{-\frac{i}{3}x^3 + ixy_1} \int_{\Gamma} e^{ixz^2 - (x^2 + y_2 - y_1)z}\, dz \tag{3.06}$$

The latter integral is easily evaluated, and we obtain finally

$$\Phi_2 = e^{i\omega(x)} \tag{3.07}$$

where

$$\omega(x) = -\frac{1}{12}x^3 + \frac{1}{2}x(y_1+y_2) + \frac{(y_2-y_1)^2}{4x} \tag{3.08}$$

As shown in Chapter 12, the quantity ω is the phase of the incident wave, and approximately equal to

$$\omega = k(R-s) \tag{3.09}$$

where R and s designate the same quantities as in Section 1. Thus the integral Φ_2 corresponds to the incident wave.

We now pass to the evaluation of the integral Φ_1. Using the integral representation (3.04) for both factors $w_1(t-y_2)$ and $w_1(t-y_1)$ and performing the integration over t, we obtain a double contour integral, in which we can introduce new variables and then perform one integration. Thus we get

$$\Phi_1 = \frac{\sqrt{x}}{2\pi i}\int_C e^{i\omega(z)}\frac{dz}{\sqrt{z}\cdot(z-x)} \tag{3.10}$$

where the contour C runs from positive imaginary infinity, intersects the real axis to the *right* of the point $z=x$, and then proceeds along the line $\operatorname{arc} z = -\pi/6$.

The residue of the integral (3.10) at the point $z=x$ according to (3.07) is equal to the quantity Φ_2. Consequently, if we denote by C' a contour, which runs similarly to C, but cuts the real axis to the *left* of the point $z=x$, we will have

$$\Phi = \Phi_1+\Phi_2 = \frac{\sqrt{x}}{2\pi i}\int_{C'} e^{i\omega(z)}\frac{dz}{\sqrt{z}\cdot(z-x)} \tag{3.11}$$

By means of these formulae we can express the function Φ approximately in terms of Fresnel's integrals. For this purpose we can use the saddlepoint method, taking into consideration, however, that the fraction $1/(z-x)$ is not a slowly-varying function. If we equate the derivative of the phase $w(z)$ to zero we come to the equation

$$z^4-2z^2(y_1+y_2)+(y_1-y_2)^2 = 0 \tag{3.12}$$

which has the roots

$$z = \pm\sqrt{y_1}\pm\sqrt{y_2} \tag{3.13}$$

Of these four roots only the largest positive one

$$z_0 = \sqrt{y_1}+\sqrt{y_2} \tag{3.14}$$

is of interest to us since it lies closest to the contour C. We will call C_0 a contour similar to C or C', but cutting the real axis at the point $z=z_0$.

Using the notation of (2.03) we put

$$x-z_0 = x-\sqrt{y_1}-\sqrt{y_2} = \xi \tag{3.15}$$

If $\xi < 0$, the contour C_0 is equivalent to C and the integral over it yiedls Φ_1. If $\xi > 0$, then the contour C_0 is equivalent to C' and the integral over it yields Φ.

Near $z = z_0$ we have

$$\omega(z) = \omega_0 - \mu^2(z-z_0)^2 \tag{3.16}$$

where

$$\omega_0 = \omega(z_0) = \frac{2}{3} y_1^{3/2} + \frac{2}{3} y_2^{3/2} \tag{3.17}$$

$$\mu^2 = \frac{\sqrt{(y_1 y_2)}}{\sqrt{y_1}+\sqrt{y_2}} \tag{3.18}$$

For an approximate evaluation of the integral

$$I = \frac{\sqrt{x}}{2\pi i} \int_{C_0} e^{i\omega(z)} \frac{dz}{\sqrt{z}\cdot(z-x)} \tag{3.19}$$

we replace the quantity $\sqrt{z}$ by the constant value $\sqrt{z_0}$ and the function $\omega(z)$ by the expression (3.16). Putting

$$z = z_0 + p\, e^{-i\frac{\pi}{4}} \tag{3.20}$$

we can integrate over p from $-\infty$ to $+\infty$. Thus we get

$$I = \sqrt{\left(\frac{x}{z_0}\right)} e^{i\omega_0} \cdot \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-\mu^2 p^2} \frac{dp}{p - \xi\, e^{i\frac{\pi}{4}}} \tag{3.21}$$

This integral is expressible in terms of Fresnel's integrals. For $\xi < 0$ and for $\xi > 0$ it has a different analytic form, namely

$$\frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-\mu^2 p^2} \frac{dp}{p-\xi\, e^{i\frac{\pi}{4}}} = \begin{cases} f(-\mu\xi) & \text{for } \xi > 0 \quad (3.22) \\ -f(-\mu\xi) & \text{for } \xi < 0 \quad (3.23) \end{cases}$$

where

$$f(\alpha) = e^{-i\alpha^2 - i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\alpha}^{\infty} e^{i\alpha^2}\, d\alpha \tag{3.24}$$

It is easy to see that

$$f(\alpha) + f(-\alpha) = e^{-i\alpha^2} \tag{3.25}$$

If we introduce the usual Fresnel Integrals

$$C+iS = \sqrt{\left(\frac{2}{\pi}\right)}\int_0^\alpha e^{i\alpha^2}\, d\alpha \tag{3.26}$$

we may write

$$f(\alpha) = \frac{1}{\sqrt{2}} e^{-i\alpha^2 - i\frac{\pi}{4}}\left[\left(\frac{1}{2}-C\right)+i\left(\frac{1}{2}-S\right)\right] \tag{3.27}$$

The asymptotic expression for $f(\alpha)$, which holds for large positive values of α, is of the form

$$f(\alpha) = \frac{1}{2\sqrt{\pi}} e^{i\frac{\pi}{4}}\left(\frac{1}{\alpha}-\frac{i}{2\alpha^3}+\ldots\right) \tag{3.28}$$

Expressing the integral I in terms of $f(\alpha)$ and remembering that this integral represents, for $\xi > 0$ the function Φ and for $\xi < 0$ the function $\Phi_1 = \Phi - \Phi_2$ defined by (3.07), we obtain finally

$$\Phi = \frac{\sqrt{x}}{\sqrt[4]{(y_1 y_2)}} e^{i\omega_0} \cdot \mu f(\mu\xi) \qquad (\text{for } \xi > 0) \tag{3.29}$$

$$\Phi = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{(y_1 y_2)}} e^{i\omega_0} \cdot \mu f(-\mu\xi) \qquad (\text{for } \xi < 0) \tag{3.30}$$

These expressions hold under the condition that both numbers y_1 and y_2 are very large (the quantity μ^2 will be of the order of the smaller of these numbers). As to the quantity ξ it may be regarded as finite or small, so that the product $\mu\xi$ may be an arbitrary number (large, finite or small). If ξ is very small (and of either sign), then both expressions for Φ practically coincide. This follows from the approximate equations

$$\frac{\mu^2 x}{\sqrt{(y_1 y_2)}} = 1 + \frac{\xi}{\sqrt{y_1}+\sqrt{y_2}} \sim 1 \tag{3.31}$$

$$\omega(x) \sim \omega_0 - \mu^2\xi^2 \tag{3.32}$$

in conjunction with the formula (3.25). For $\xi = 0$ the two expressions for Φ coincide exactly.

4. *Evaluation of the Integral Ψ*

We now proceed to the derivation of approximate formulae for the integral Ψ. We have to find the approximate value of the integral for the same case for which we calculated the integral Φ namely when the quantities $\sqrt{y}$, $\sqrt{y_2}$ (and consequently μ^2) are very large, while the quantity

$\xi = x-\sqrt{y_1}-\sqrt{y_2}$ is finite. Under these conditions the principal part of the contour of integration is that part on which the variable t is finite. Now, for finite t, and for y_1 and y_2 large, the product of the functions two w_1 and the exponential in the integrand in (2.06) is equal to

$$e^{ixt}\, w_1(t-y_1)w_1(t-y_2) = \frac{i}{\sqrt[4]{(y_1y_2)}}\, e^{i\omega_0}\, e^{i\xi t}\left[1+\frac{it^2}{4\mu^2}+0\left(\frac{1}{\mu^4}\right)\right] \tag{4.01}$$

where for the sake of brevity we used the notation of (3.17).

Inserting this expression into the integral Ψ, we obtain

$$\Psi = -\frac{\sqrt{x}}{\sqrt[4]{(y_1y_2)}}\, e^{i\omega_0}\left[g(\xi)-\frac{i}{4\mu^2}g''(\xi)+0\left(\frac{1}{\mu^4}\right)\right] \tag{4.02}$$

where

$$g(\xi) = \frac{1}{\sqrt{\pi}}\, e^{i\frac{\pi}{4}}\left[\frac{i}{2}\int_{\infty e^{i\frac{2}{3}\pi}}^{0} e^{i\xi t}\frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)}\,dt + \right.$$
$$\left. +\int_0^{\infty} e^{i\xi t}\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}\,dt\right] \tag{4.03}$$

Making use of the properties of the Airy function (1.15), we see immediately, that if

$$t = t'\, e^{i\frac{2}{3}\pi} \tag{4.04}$$

it follows

$$\frac{i}{2}\,\frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} = \frac{v'(t')-q\,e^{i\frac{2}{3}\pi}\,v(t')}{w_2'(t')-q\,e^{i\frac{2}{3}\pi}\,w_2(t')} \tag{4.05}$$

Substitution of (4.04) reduces the first integral in (4.03) to an integral over the positive real axis. Omitting the prime on t we get

$$g(\xi) = e^{-i\frac{\pi}{12}}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{-\frac{\xi t}{2}(\sqrt{3}+i)}\cdot\frac{v'(t)-q\,e^{i\frac{2}{3}\pi}\,v(t)}{w_2'(t)-q\,e^{i\frac{2}{3}\pi}\cdot w_2(t)}\,dt +$$
$$+e^{i\frac{\pi}{4}}\cdot\frac{1}{\sqrt{\pi}}\int_0^{\infty} e^{i\xi t}\cdot\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}\,dt \tag{4.06}$$

As t increases the function $v(t)$ in the numerator decreases rapidly while the functions $w_1(t)$ and $w_2(t)$ in the denominator increase rapidly. Therefore,

both integrals converge very rapidly, and may be evaluated by numerical methods. The function $g(\xi)$ may be developed in a Taylor series in powers of ξ; the coefficients of this series may also be evaluated in the above manner. For large, positive values of ξ the function $g(\xi)$ has an asymptotic expression consisting of a single term

$$g(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \cdot \frac{1}{\xi} \tag{4.07}$$

not depending on q. The remainder is of the order $e^{i\xi t_1}$ where t_1 is the first root of the equation

$$w_1'(t) - q w_1(t) = 0 \tag{4.08}$$

For large negative values of ξ the asymptotic expression for $g(\xi)$ has the form

$$g(\xi) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{\pi}} \cdot \frac{1}{\xi} + \frac{\sqrt{-\xi}}{2} \cdot \frac{q + i\frac{\xi}{2}}{q - i\frac{\xi}{2}} e^{-\frac{i}{12}\xi^2} \tag{4.09}$$

Before inserting this expression in the formula (4.02) for Ψ one has to remember, that this formula applies only when the correction term, which contains μ^2 in the denominator, is small as compared to the main term. In order that both expressions (4.02) and (4.09) be applicable the condition

$$1 \ll \xi^2 \ll \mu^2 \qquad (\xi < 0) \tag{4.10}$$

must be satisfied.

5. *The Attenuation Factor in the Region of the Shadow-Cone*

In the preceding sections we found approximate expressions for the integrals Φ and Ψ, whose sum yields the attenuation factor $V(x, y_1, y_2, q)$. Forming this sum, we obtain for $\xi \geqslant 0$

$$V = \frac{\sqrt{x}}{\sqrt[4]{(y_1 y_2)}} e^{i\omega_0} \left[\mu f(\mu\xi) - g(\xi) + \frac{i}{4\mu^2} g''(\xi) \right] \tag{5.01}$$

and for $\xi \leqslant 0$

$$V = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{(y_1 y_2)}} e^{i\omega_0} \left[\mu f(-\mu\xi) + g(\xi) - \frac{i}{4\mu^2} g''(\xi) \right] \tag{5.02}$$

These expressions are valid provided the parameter μ, determined from

the equation

$$\mu^2 = \frac{\sqrt{(y_1 y_2)}}{\sqrt{y_1}+\sqrt{y_2}}. \tag{5.03}$$

is very large, while the quantity

$$\xi = x-\sqrt{y_1}-\sqrt{y_2} \tag{5.04}$$

is finite or small.

Let us return to the geometrical meaning of these quantities. According to the formulae (1.03) to (1.05) we have

$$\mu^2 = \sqrt[6]{\left(\frac{k}{2a^2}\right)} \cdot \frac{\sqrt{(h_1 h_2)}}{\sqrt{h_1}+\sqrt{h_2}} \tag{5.05}$$

$$\xi = \sqrt[3]{\frac{k}{2a^2}}\left[s-\sqrt{(2ah_1)}-\sqrt{(2ah_2)}\right] \tag{5.06}$$

Thus large values of μ correspond to short wavelengths and to relatively large distances from the surface of the body (the distances should still be small as compared to the radii of curvature of the body). The quantity ξ is proportional to the distance from the geometrical boundary of the shadow (the shadow cone), the distance being taken along (more exactly, parallel to) the surface of the body. For $\xi < 0$ the quantity $\mu^2\xi^2$ is approximately equal to the phase difference between the incident and the reflected wave (equation (3.32)). The value $\xi = 0$ corresponds to the boundary of the shadow, positive values of ξ correspond to the shadow, and negative values to the illuminated region.

Our formulae give the transition between light and shadow at relatively large distances from the surface of the body. Since the functions f and g and their derivatives with respect to their arguments are, for finite values of these arguments, of the order unity, it follows that the principal term in (5.01) will be the term $\mu f(\mu\, \xi)$. This term is proportional to the Fresnel integral. It is a rapidly-varying function of ξ, since the argument of the Fresnel integral is $\mu\xi$ where μ is a large number. Thus the main term in the expression for V yields the Fresnel diffraction. But on this diffraction pattern there is superposed a background represented by the function $g(\xi)$ which is slowly varying in comparison with the function $f(\mu\xi)$ giving the main term. This background depends on the material of the diffracting body (since $g(\xi)$ depends on q), while the Fresnel term is independent of it. As one goes further away from the shadow cone in both directions, the expressions derived here for the attenuation factor agree with the

previously derived formulae for the shadow and the illuminated region. Let us verify this. In the shadow region we must have an exponential amplitude decay and in the illuminated region the reflection formula. Since in the formula (5.01) and in the asymptotic expression (4.07) for $g(\xi)$, we did not take into account terms which decrease exponentially for large positive ξ (these terms being smaller than the error of the asymptotic expressions), in our approximation we should get the value zero in the shadow region. Indeed, from the asymptotic expression (3.28) for the Fresnel function $f(\alpha)$ it follows:

$$\mu f(\mu\xi) = \frac{1}{2\sqrt{\pi}} \cdot e^{i\frac{\pi}{4}} \cdot \left(\frac{1}{\xi} - \frac{i}{2\mu^2\xi^3}\right) \tag{5.07}$$

On the other hand, formula (4.07) gives

$$g(\xi) - \frac{i}{4\mu^2} g''(\xi) = \frac{1}{2\sqrt{\pi}} e^{i\frac{\pi}{4}} \cdot \left(\frac{1}{\xi} - \frac{i}{2\mu^2\xi^3}\right) \tag{5.08}$$

that is, the same expression. Thus for large positive ξ the expression (5.01) for V actually vanishes in our approximation.

We now consider large negative values of ξ. In the formula (5.02) the first term of the asymptotic expression (4.09) for $g(\xi)$ cancels with $\mu f(-\mu\xi)$, the second term (which contains the exponential function) gives

$$V = e^{i\omega(x)} - \frac{\sqrt{x}}{\sqrt[4]{(y_1 y_2)}} \cdot e^{i\omega_0} \cdot \frac{\sqrt{(-\xi)}}{2} \cdot \frac{q + i\dfrac{\xi}{2}}{q - i\dfrac{\xi}{2}} \cdot e^{-\frac{i}{12}\xi^3} \tag{5.09}$$

On the other hand, in the illuminated region the reflection formula

$$V = e^{i\omega} \cdot \left[1 - \frac{q - ip}{q + ip} \cdot \sqrt{\left(\frac{p}{p + p_1}\right)} \cdot e^{2ip_1 p^2}\right] \tag{5.10}$$

holds, as shown in Chapter 12 (the formula (4.31) of that chapter). We have here $\omega = \omega(x)$, and the quantity p (which is proportional to the cosine of the angle of incidence) is determined from the equation:

$$\sqrt{(y_1 + p^2)} + \sqrt{(y_2 + p^2)} = 2p + x \tag{5.11}$$

while the quantity p_1 is equal to

$$p_1 = 2p + x - \frac{1}{x}(y_1 + y_2) \tag{5.12}$$

In the approximation in which formula (5.09) holds, we have

$$p = -\frac{\xi}{2}+\frac{\xi^2}{16\mu^2} \sim -\frac{\xi}{2} \tag{5.13}$$

$$p_1 = 2\mu^2+\xi-\frac{2\mu^2\xi}{x} \sim 2\mu^2 \tag{5.14}$$

Using these approximate equalities, one can easily verify that the equation (5.09) is an approximate form of the reflection formula (5.10).

Thus, the formulae (5.01) and (5.04), which were derived for the region near the shadow cone, agree in this region with the formulae which hold in the regions adjoining this shadow cone on both sides, and which were derived in our previous papers.

In conclusion, we make the following remarks about the formulae derived above.

Our initial formula for V, as well as our approximate formulae admit of transition to the case of a plane wave provided a suitable change in the expressions for the phase of the incident wave is made. This transition consists in letting x and $\sqrt{y_2}$ tend to infinity, while keeping their difference finite. Now, in the case of a plane wave our initial formulae are valid not only for a sphere, but also for a body of arbitrary shape, as shown in Chapters 5 and 6. Therefore, we may regard the approximate formulae derived above, containing the Fresnel integrals, as valid also for a body of arbitrary shape. It is also very probable that the diffraction pattern found here (a Fresnel diffraction, superposed on a background) is valid, at least qualitatively, also at large distances from the body, and it is to be expected that the background becomes fainter as the distance from the body increases.

CHAPTER 8

GENERALIZATION OF THE REFLECTION FORMULAE TO THE CASE OF REFLECTION OF AN ARBITRARY WAVE FROM A SURFACE OF ARBITRARY FORM†

Abstract — On the basis of the Fresnel formulae and the laws of Hamilton's ray optics expressions for the electromagnetic field of an arbitrary wave reflected from a a surface of arbitrary form are derived. A correction for the broadening of the pencil of rays after reflection is taken into account. In this derivation the tensor form of differential geometry is used for the reflecting surface. The Gaussian parameters of the surface at the point of reflection and the phase of the reflected wave are considered as curvilinear coordinates. For the particular case of a spherical wave reflected from a sphere, the formulae obtained are compared with those derived from diffraction theory.

In Chapter 6 a reflection formula, taking into account the broadening of the bundle of reflected rays was derived for the case of a plane wave reflected from a surface of an arbitrary form. This formula was then compared with diffraction formulae valid in the penumbra region.

In this chapter, a reflection formula is derived for the case of reflection of an arbitrary (not necessarily plane) wave. Our calculations are based on the application of the Fresnel reflection laws, established about 1820, and the laws of ray optics established by Hamilton about 1830. Our resnlts cannot therefore be considered new in principle. However, since the Fresnel formulae are applied to the electro-magnetic field, and since the laws of the ray optics are formulated with aid of differential geometry in its tensor from (which leads to essential simplifications), our results may prove to be useful for practical applications. For the convenience of the reader unfamiliar with the tensor form of differential geometry, we present a summary (in Section 2) of the necessary formulae.

† Fock, 1950.

1. *Fresnel formulae*

Let the field of an incident wave be represented by

$$\mathbf{E}^0 e^{ik\psi}, \quad \mathbf{H}^0 e^{ik\psi} \tag{1.01}$$

where $\mathbf{E}^0$ and $\mathbf{H}^0$ denotes its amplitude, and ψ is the phase expressed in units of length, and

$$(\text{grad}\, \psi)^2 = 1 \tag{1.02}$$

For a plane wave, the amplitudes $\mathbf{E}^0$ and $\mathbf{H}^0$ are constant; in the general case we shall consider the components of vectors $\mathbf{E}^0$ and $\mathbf{H}^0$ as slowly varying functions of coordinates. In the following, $\mathbf{E}^0$ and $\mathbf{H}^0$ are understood to be the values of the amplitude of the field on the surface of a reflecting body. The corresponding values for the reflected wave will be designated by $\mathbf{E}^*$ and $\mathbf{H}^*$.

Further, let $\mathbf{a}(a_x, a_y, a_z)$ be a unit vector in the direction of an incident ray, $\mathbf{a}^*(a_x^*, a_y^*, a_z^*)$ – a unit vector in the direction of the reflected ray, and $\mathbf{n}(n_x, n_y, n_z)$ – the unit vector of a normal to the surface of the body at the point of reflection. According to the law of reflection, the values $\mathbf{a}^*$, $\mathbf{a}$ and $\mathbf{n}$ are connected by the relation:

$$\mathbf{a}^* = \mathbf{a} - 2\mathbf{n}(\mathbf{a}\cdot\mathbf{n}) \tag{1.03}$$

and

$$\mathbf{a}^*\cdot\mathbf{n} = -\mathbf{a}\cdot\mathbf{n} = \cos\vartheta \tag{1.04}$$

where ϑ is the angle of incidence. The quantities $\mathbf{a}$ and $\mathbf{a}^*$ are proportional to the gradient of the phase of the incident and the reflected wave respectively. Neglecting the variation of the amplitude over one wavelength, we obtain from the Maxwell equations in vacuum

$$[\mathbf{a}\times\mathbf{E}^0] = \mathbf{H}^0; \qquad \mathbf{a}\cdot\mathbf{E}^0 = 0 \tag{1.05}$$

whence

$$[\mathbf{a}\times\mathbf{H}^0] = -\mathbf{E}^0; \qquad \mathbf{a}\cdot\mathbf{H}^0 = 0 \tag{1.06}$$

and similarly for the reflected wave

$$[\mathbf{a}^*\times\mathbf{E}^*] = \mathbf{H}^*; \qquad \mathbf{a}^*\cdot\mathbf{E}^* = 0 \tag{1.07}$$

$$[\mathbf{a}^*\times\mathbf{H}^*] = -\mathbf{E}^*; \qquad \mathbf{a}^*\cdot\mathbf{H}^* = 0 \tag{1.08}$$

We designate by μ the magnetic permeability, and by

$$\eta = \varepsilon + i4\pi\sigma/\omega \tag{1.09}$$

the complex dielectric constant of the reflecting body, and introduce the Fresnel coefficients

$$N = \frac{\eta \cos \vartheta - \sqrt{(\mu\eta - \sin^2 \vartheta)}}{\eta \cos \vartheta + \sqrt{(\mu\eta - \sin^2 \vartheta)}} \tag{1.10}$$

$$M = \frac{\mu \cos \vartheta - \sqrt{(\mu\eta - \sin^2 \vartheta)}}{\mu \cos \vartheta + \sqrt{(\mu\eta - \sin^2 \vartheta)}} \tag{1.11}$$

Then Fresnel formulae, which establish the relation between the amplitudes of the incident and the reflected wave, can be written in the form

$$(\mathbf{n} \cdot \mathbf{E}^*) = N(\mathbf{n} \cdot \mathbf{E}^0) \tag{1.12}$$

$$(\mathbf{n} \cdot \mathbf{H}^*) = M(\mathbf{n} \cdot \mathbf{H}^0) \tag{1.13}$$

The amplitudes of the transmitted wave (which penetrates the substance of the body) are of no interest to us, and we do not write out the corresponding formulae.

Equations (1.07), (1.12) and (1.13) can be solved for the vectors $\mathbf{E}^*$ and $\mathbf{H}^*$. Introducing the notations

$$\mathbf{n} \cdot \mathbf{E}^0 = E_n^0; \qquad \mathbf{n} \cdot \mathbf{H}^0 = H_n^0 \tag{1.14}$$

and expressing $\mathbf{a}^*$, according to (1.03), in terms of $\mathbf{a}$, we have

$$\sin^2 \vartheta\, \mathbf{E}^* = -NE_n^0(\mathbf{n} \cos 2\vartheta + \mathbf{a} \cos \vartheta) + MH_n^0[\mathbf{n} \times \mathbf{a}] \tag{1.15}$$

$$\sin^2 \vartheta\, \mathbf{H}^* = -MH_n^0(\mathbf{n} \cos 2\vartheta + \mathbf{a} \cos \vartheta) - NE_n^0[\mathbf{n} \times \mathbf{a}] \tag{1.16}$$

The latter formulae can be written in a somewhat different form, if we introduce instead of $\mathbf{a}$ and $\mathbf{a}^*$ a vector tangent to the surface

$$\mathbf{b} = \mathbf{a} + \mathbf{n} \cos \vartheta = \mathbf{a}^* - \mathbf{n} \cos \vartheta \tag{1.17}$$

the square of which

$$\mathbf{b}^2 = \sin^2 \vartheta \tag{1.18}$$

We have

$$\sin^2 \vartheta\, \mathbf{E}^* = NE_n^0(\mathbf{n} \sin^2 \vartheta - \mathbf{b} \cos \vartheta) + MH_n^0[\mathbf{n} \times \mathbf{b}] \tag{1.19}$$

$$\sin^2 \vartheta\, \mathbf{H}^* = MH_n^0(\mathbf{n} \sin^2 \vartheta - \mathbf{b} \cos \vartheta) - NE_n^0[\mathbf{n} \times \mathbf{b}] \tag{1.20}$$

These are the amplitude values of a wave reflected from the surface of a body as derived from the Fresnel formulae.

2. *Differential Geometry of the Reflecting Surface*

Let the equation of the reflecting surface in parametric form be

$$x = x_0(u, v); \qquad y = y_0(u, v); \qquad z = z_0(u, v) \tag{2.01}$$

where u, v are Gaussian coordinate parameters (curvilinear coordinates on the surface). Putting

$$g_{uv} = \frac{\partial x_0}{\partial u}\frac{\partial x_0}{\partial v}+\frac{\partial y_0}{\partial u}\frac{\partial y_0}{\partial v}+\frac{\partial z_0}{\partial u}\frac{\partial z_0}{\partial v} \tag{2.02}$$

and defining similarly g_{uu} and g_{vv}, we write the square of the arc element on the surface in the form

$$d\sigma^2 = g_{uu}\,du^2+2g_{uv}\,du\,dv+g_{vv}\,dv^2 \tag{2.03}$$

or shorter

$$d\sigma^2 = \sum_{u,v} g_{uv}\,du\,dv \tag{2.04}$$

We shall use the notations for covariant and contravariant components of vectors and tensors and the rules for raising and lowering the indices with the aid of the "metric" tensor g_{uv} appearing in (2.04).
If we put

$$g = g_{uu}g_{vv}-(g_{uv})^2 \tag{2.05}$$

then the contravariant components of the metric tensor will be equal to

$$g^{uu} = \frac{g_{vv}}{g}; \qquad g^{uv} = -\frac{g_{uv}}{g}; \qquad g^{vv} = \frac{g_{uu}}{g}. \tag{2.06}$$

The quantities (2.06) constitute a tensor which is also called the inverse of the tensor g_{uv}. The element of the surface will be written in the form

$$dS = \sqrt{g}\cdot du\,dv \tag{2.07}$$

In the following we shall deal with covariant differentiation on the surface. For this we put

$$\frac{\partial x_0}{\partial w}\frac{\partial^2 x_0}{\partial u\,\partial v}+\frac{\partial y_0}{\partial w}\frac{\partial^2 y_0}{\partial u\,\partial v}+\frac{\partial z_0}{\partial w}\frac{\partial^2 z_0}{\partial u\,\partial v} = [uv, w] \tag{2.08}$$

where the combination u, v, can be replaced by u, u or v, v and the letter w may have the values u, v. The quantities $[uv, w]$ called Christoffel's symbols, can be expressed by derivatives of g_{uv}, namely

$$[uv, w] = \frac{1}{2}\left(\frac{\partial g_{uw}}{\partial v}+\frac{\partial g_{vw}}{\partial u}-\frac{\partial g_{uv}}{\partial w}\right). \tag{2.09}$$

In our case there are six Christoffel's symbols — three quantities of the

form

$$[uu, u] = \frac{1}{2}\frac{\partial g_{uu}}{\partial u}; \qquad [uv, u] = \frac{1}{2}\frac{\partial g_{uu}}{\partial v}$$
$$[uu, v] = \frac{\partial g_{uv}}{\partial u} - \frac{1}{2}\frac{\partial g_{uu}}{\partial v} \tag{2.10}$$

and the other three quantities obtained from the preceding ones by interchanging u with v, With their aid we form "tensorial parameters" (or "Christoffel's symbols of the second kind"), i.e. the quantities

$$\Gamma^p_{qr} = g^{pu}[qr, u] + g^{pv}[qr, v] \tag{2.11}$$

where each of the letters p, q and r may take the values u, v.

Let $f(u, v)$ be a certain function of a point on the surface. The covariant components of its gradient on the surface will be equal to

$$f_u = \frac{\partial f}{\partial u}; \qquad f_v = \frac{\partial f}{\partial v} \tag{2.12}$$

and the contravariant components will be

$$f^u = g^{uu}f_u + g^{uv}f_v; \qquad f^v = g^{uv}f_u + g^{vv}f_v \tag{2.13}$$

with the square of the gradient being equal to

$$f_u f^u + f_v f^v = g^{uu}\left(\frac{\partial f}{\partial u}\right)^2 + 2g^{uv}\frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + g^{vv}\left(\frac{\partial f}{\partial v}\right)^2 \tag{2.14}$$

The square of the gradient is a scalar, i.e. it is not dependent on the choice of the coordinate parameters u, v.

The second covariant derivative of the function $f(u, v)$ differs from the usual second derivative by terms linear in the first derivatives of the same function. We have

$$f_{uu} = \frac{\partial^2 f}{\partial u^2} - \Gamma^u_{uu}\frac{\partial f}{\partial u} - \Gamma^v_{uu}\frac{\partial f}{\partial v}$$
$$f_{uv} = \frac{\partial^2 f}{\partial u\,\partial v} - \Gamma^u_{uv}\frac{\partial f}{\partial u} - \Gamma^v_{uv}\frac{\partial f}{\partial v} \tag{2.15}$$
$$f_{vv} = \frac{\partial^2 f}{\partial v^2} - \Gamma^u_{vv}\frac{\partial f}{\partial u} - \Gamma^v_{vv}\frac{\partial f}{\partial v}$$

It can be shown that the set of quantities f_{uu}, f_{uv} and f_{vv} represents a symmetrical tensor, and the expression

$$f_{uu}\,\mathrm{d}u^2 + 2f_{uv}\,\mathrm{d}u\,\mathrm{d}v + f_{vv}\,\mathrm{d}v^2 \tag{2.16}$$

does not depend on the choice of the coordinates u, v.

Let us now consider the formulae for the components of the vector of the normal to the surface and also their derivatives with respect to the parameters u and v; these derivatives are connected with the radii of curvature of the normal sections of the surface. We have

$$\sqrt{g}\cdot n_x = \frac{\partial y_0}{\partial u}\frac{\partial z_0}{\partial v} - \frac{\partial y_0}{\partial v}\frac{\partial z_0}{\partial u} \qquad \text{etc.} \tag{2.17}$$

where the letters "etc." mean two similar expressions, obtained by a cyclic permutation of the letters (x, y, z).

It is obvious that

$$n_x\frac{\partial x_0}{\partial u}+n_y\frac{\partial y_0}{\partial u}+n_z\frac{\partial z_0}{\partial u} = 0; \qquad n_x\frac{\partial x_0}{\partial v}+n_y\frac{\partial y_0}{\partial v}+n_z\frac{\partial z_0}{\partial v} = 0 \tag{2.18}$$

We put

$$\begin{aligned} G_{uu} &= n_x\frac{\partial^2 x_0}{\partial u^2}+n_y\frac{\partial^2 y_0}{\partial u^2}+n_z\frac{\partial^2 z_0}{\partial u^2} \\ G_{uv} &= n_x\frac{\partial^2 x_0}{\partial u\,\partial v}+n_y\frac{\partial^2 y_0}{\partial u\,\partial v}+n_z\frac{\partial^2 z_0}{\partial u\,\partial v} \\ G_{vv} &= n_x\frac{\partial^2 x_0}{\partial v^2}+n_y\frac{\partial^2 y_0}{\partial v^2}+n_z\frac{\partial^2 z_0}{\partial v^2} \end{aligned} \tag{2.19}$$

With help of the equations (2.18), we can replace here the usual second derivatives of x_0, y_0, z_0 by the covariant ones. Indeed, by assuming in (2.15) successively $f=x_0$; $f=y_0$; $f=z_0$ multiplying by n_x, n_y, n_z, and adding, we obtain on the left-hand side a linear combination of covariant second derivatives, and on the right, the expressions (2.19) (on the right-hand side, the terms involving the first derivatives will cancel by virtue of (2.18)). Hence, it follows that the quantities G_{uu}, G_{uv} and G_{vv} represent a tensor, which will obviously be symmetrical. Using the same equations (2.18) the quantities G_{uv}, taken with the opposite sign, can be written in the form

$$-G_{uv} = \frac{\partial n_x}{\partial u}\frac{\partial x_0}{\partial v}+\frac{\partial n_y}{\partial u}\frac{\partial y_0}{\partial v}+\frac{\partial n_z}{\partial u}\frac{\partial z_0}{\partial v} \tag{2.20}$$

Hence it follows that

$$-\sum_{u,v} G_{uv}\,\mathrm{d}u\,\mathrm{d}v = \mathrm{d}n_x\,\mathrm{d}x_0+\mathrm{d}n_y\,\mathrm{d}y_0+\mathrm{d}n_z\,\mathrm{d}z_0 \tag{2.21}$$

Putting

$$\mathrm{d}n_x = (\mathrm{d}x_0/R)+\delta n_x \qquad \text{etc.} \tag{2.22}$$

where the infinitely small vector $\delta\mathbf{n}$ is perpendicular to the vector $\mathbf{n}$ of the normal and to the displacement vector (dx_0, dy_0, dz_0), we obtain

$$-\sum_{u,v} G_{u,v}\, du\, dv = d\sigma^2/R \tag{2.23}$$

where $d\sigma^2$ is the square of the displacement vector given by expression (2.03). Relations (2.22) show that R is the radius of curvature of the curve formed by the section of the surface with the plane containing the normal and the displacement vector. Thus, the formula (2.23) yields an expression for the radius of curvature, R, for a given direction of the plane of the normal section.

Solving equations (2.20) for the derivatives of n_x, n_y and n_z with respect to u, v we obtain

$$\begin{aligned} \frac{\partial n_x}{\partial u} &= -G_u^u \frac{\partial x_0}{\partial u} - G_u^v \frac{\partial x_0}{\partial v} \\ \frac{\partial n_x}{\partial v} &= -G_v^u \frac{\partial x_0}{\partial u} - G_v^v \frac{\partial x_0}{\partial v} \end{aligned} \tag{2.24}$$

where the quantities G_v^u are obtained from G_{uv} by means of formulae similar to (2.13).

Designating the principal radii of curvature of the normal section by R_1 and R_2, we have

$$K = 1/(R_1R_2) = G_u^uG_v^v - G_v^uG_u^v \tag{2.25}$$

$$(1/R_1)+(1/R_2) = -G = -G_u^u - G_v^v \tag{2.26}$$

The quantity K is the Gaussian curvature of the surface.

3. *Cross-section of a Bundle of Reflected Rays*

Fresnel formulae give the amplitude of the reflected wave on the surface of a body. In order to find the amplitude values of the reflected wave at a certain distance from the surface, it is necessary to have formulae for the cross-section of the bundle, reflected by a surface element dS of the body, and which after reflection has travelled a given distance s. In Chapter 6 we have derived such formulae when the incident wave is plane. In the present chapter we shall derive them for the general case of an arbitrary incident wave.

According to the law of reflection written in the form (1.17), unit vectors $\mathbf{a}$ and $\mathbf{a}^*$, characterizing the direction of the incident and reflected rays, are expressed in terms of a vector $\mathbf{b}$ tangential to the surface, by means

of the formulae:

$$\mathbf{a} = \mathbf{b}-\mathbf{n}\cos\vartheta \tag{3.01}$$

$$\mathbf{a}^* = \mathbf{b}+\mathbf{n}\cos\vartheta \tag{3.02}$$

where

$$\mathbf{n}\cdot\mathbf{b} = 0; \qquad \mathbf{b}^2 = \sin^2\vartheta \tag{3.03}$$

Let the value of the phase ψ of the incident wave at a point (u, v) on the surface of the body be denoted by $\omega(u, v)$. Since the vector $\mathbf{a}$ is the gradient of the phase ψ, the tangential components of the vector $\mathbf{a}$, (which, by virtue of (3.01), are equal to the tangential components of the vector $\mathbf{b}$) can be expressed in terms of the derivatives of ω with respect to u and v. These derivatives are, in turn, expressible in terms of the components of the vector $\mathbf{b}$. We have

$$\begin{aligned}\frac{\partial\omega}{\partial u} &= \omega_u = b_x\frac{\partial x_0}{\partial u}+b_y\frac{\partial y_0}{\partial u}+b_z\frac{\partial z_0}{\partial u}\\ \frac{\partial\omega}{\partial v} &= \omega_v = b_x\frac{\partial x_0}{\partial v}+b_y\frac{\partial y_0}{\partial v}+b_z\frac{\partial z_0}{\partial v}\end{aligned} \tag{3.04}$$

Combining this with the first equation (3.03), we can solve these three equations for b_x, b_y, b_z. We obtain

$$b_x = \omega^u\frac{\partial x_0}{\partial u}+\omega^v\frac{\partial x_0}{\partial v} \qquad \text{etc.} \tag{3.05}$$

where the quantities ω^u, ω^v are connected with the derivatives ω_u, ω_v by relations similar to (2.13).

The second equation in (3.03) can be written in the form

$$\sum_{u,v} g_{uv}\omega^u\omega^v = \sum_{u,v} g^{uv}\omega_u\omega_v = \omega_u\omega^u+\omega_v\omega^v = \sin^2\vartheta \tag{3.06}$$

Thus, the angle of incidence ϑ is expressed directly by ω_u, ω_v.

Let us consider the equations

$$x = x_0+sa_x^* \qquad \text{etc.} \tag{3.07}$$

which can be written in the form

$$x = x_0+sb_x+s\cos\vartheta\, n_x \qquad \text{etc.} \tag{3.08}$$

All quantities on the right-hand side of these equations, except s, represent certain known functions of the point (u, v) on the surface. Considering (u, v) as constant, and varying s, we obtain the equation of the ray

reflected at the point (u, v). The parameter s is obviously the path the ray travelled after reflection. Since the phase of the incident wave at the point of reflection is $\omega(u, v)$, the phase χ of the reflected wave will then be equal to

$$\chi = s+\omega(u, v) \tag{3.09}$$

Expressing s in (3.07) in terms of χ, we obtain

$$\begin{aligned} x &= x_0+(\chi-\omega)a_x^* \\ y &= y_0+(\chi-\omega)a_y^* \\ z &= z_0+(\chi-\omega)a_z^* \end{aligned} \tag{3.10}$$

With a constant χ the formulae (3.10) represent parametric equations of the wave surface for the reflected wave.

If in formulae (3.10) we vary the values u, v within the limits $(u, u+\mathrm{d}u)$, $(v, v+\mathrm{d}v)$, we obtain a certain area of the wave surface. This area can be considered as a section by the wave surface of the bundle of reflected rays, reflected from a surface element $\mathrm{d}S = \sqrt{g}\cdot\mathrm{d}u\,\mathrm{d}v$ of the reflecting body. Since the rays are perpendicular to the wave surface, this cross-section will be normal. Designating its area by $D(s)\,\mathrm{d}S$ we have

$$D(s)\,\mathrm{d}S = \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \mathrm{d}u\,\mathrm{d}v \tag{3.11}$$

whence

$$D(s) = \frac{1}{\sqrt{g}} \begin{vmatrix} a_x^* & a_y^* & a_z^* \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \tag{3.12}$$

In these formulae, the quantities $\partial x/\partial u$ etc. denote derivatives of expressions (3.10), taken for a constant χ. The value of the determinants, however will not change, if they are replaced by derivatives for constant s (as was done in Chapter 6). We have

$$\left(\frac{\partial x}{\partial u}\right)_\chi = \left(\frac{\partial x}{\partial u}\right)_s - \omega_u a_x^* \quad \text{etc.} \tag{3.13}$$

and, as a result of such a replacement, the second and third lines of the determinant will change by amounts proportional to the elements of the first line. Geometrically, this means that the section of the bundle by any surface (for example, by the surface s=constant) being projected on a plane perpendicular to the reflected ray, will give the normal cross-section of the bundle.

4. *Calculation of the Determinant*

A direct calculation of the determinant (3.12) involves intricate computations. These computations may, however, be considerably simplified, if in the vectors contained in the first, second and third line of the determinant, one goes over from components along the axes x, y, z to the components along two tangential directions and the direction of the normal to the reflecting surface.

Suppose we have a determinant

$$\Delta = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{4.01}$$

We put

$$\begin{aligned} A_u &= A_x\frac{\partial x_0}{\partial u}+A_y\frac{\partial y_0}{\partial u}+A_z\frac{\partial z_0}{\partial u} \\ A_v &= A_x\frac{\partial x_0}{\partial v}+A_y\frac{\partial y_0}{\partial v}+A_z\frac{\partial z_0}{\partial v} \\ A_n &= A_x n_x+A_y n_y+A_z n_z \end{aligned} \tag{4.02}$$

whence conversely

$$\begin{aligned} A_x &= A^u\frac{\partial x_0}{\partial u}+A^v\frac{\partial x_0}{\partial v}+A_n n_x \\ A_y &= A^u\frac{\partial y_0}{\partial u}+A^v\frac{\partial y_0}{\partial v}+A_n n_y \\ A_z &= A^u\frac{\partial z_0}{\partial u}+A^v\frac{\partial z_0}{\partial v}+A_n n_z \end{aligned} \tag{4.03}$$

Here A^u and A^v are connected with A_u and A_v by formulae similar to (2.13). A similar transformation can be applied to the two other vectors, **B** and **C**, involved in the determinant.

We have then

$$\Delta = \frac{1}{\sqrt{g}}\begin{vmatrix} A_u & A_v & A_n \\ B_u & B_v & B_n \\ C_u & C_v & C_n \end{vmatrix} \tag{4.04}$$

and also

$$\Delta = \sqrt{g}\begin{vmatrix} A^u & A^v & A_n \\ B^u & B^v & B_n \\ C^u & C^v & C_n \end{vmatrix} \tag{4.05}$$

In order to apply these formulae to the calculation of the determinant (3.12), we must put

$$\begin{aligned} A_x &= a_x^*, & A_y &= a_y^*, & A_z &= a_z^* \\ B_x &= \frac{\partial x}{\partial u}, & B_y &= \frac{\partial y}{\partial u}, & B_z &= \frac{\partial z}{\partial u} \\ C_x &= \frac{\partial x}{\partial v}, & C_y &= \frac{\partial y}{\partial v} & C_z &= \frac{\partial z}{\partial v} \end{aligned} \tag{4.06}$$

By virtue of (3.02) and (3.03), we obtain

$$A_u = \omega_u; \qquad A_v = \omega_v; \qquad A_n = \cos\vartheta \tag{4.07}$$

The calculation of the new components of the vectors **B** and **C** is considerably more complicated. We have

$$B_x = \frac{\partial x_0}{\partial u} - \omega_u a_x^* + s\frac{\partial a_x^*}{\partial u} \tag{4.08}$$

and using (1.17)

$$B_x = \frac{\partial x_0}{\partial u} - \omega_u a_x^* + s\left(\frac{\partial b_x}{\partial u} + n_x\frac{\partial(\cos\vartheta)}{\partial u} + \cos\vartheta\frac{\partial n_x}{\partial u}\right) \tag{4.09}$$

According to formula (2.20), we have

$$\frac{\partial n_x}{\partial u}\frac{\partial x_0}{\partial v} + \frac{\partial n_y}{\partial u}\frac{\partial y_0}{\partial v} + \frac{\partial n_z}{\partial u}\frac{\partial z_0}{\partial v} = -G_{uv} \tag{4.10}$$

This expression is symmetrical with respect to u, v.

Now we calculate the quantity

$$b_{uv} = \frac{\partial b_x}{\partial u}\frac{\partial x_0}{\partial v} + \frac{\partial b_y}{\partial u}\frac{\partial y_0}{\partial v} + \frac{\partial b_z}{\partial u}\frac{\partial z_0}{\partial v} \tag{4.11}$$

If we use formulae (3.04) this quantity can be written in the form

$$b_{uv} = \frac{\partial^2\omega}{\partial u\,\partial v} - \left(b_x\frac{\partial^2 x_0}{\partial u\,\partial v} + b_y\frac{\partial^2 y_0}{\partial u\,\partial v} + b_z\frac{\partial^2 z_0}{\partial u\,\partial v}\right) \tag{4.12}$$

whence it is evident that b_{uv} is likewise symmetrical with respect to u, v. Replacing b_x, b_y, b_z by the expressions (3.05) and using (2.08), we can

write this quantity in the form

$$b_{uv} = \frac{\partial^2\omega}{\partial u\,\partial v} - \omega^u[uv, u] - \omega^v[uv, v] \tag{4.13}$$

Introducing according to (2.11) the "tensor parameters" Γ^p_{qr} we can also write

$$b_{uv} = \frac{\partial^2\omega}{\partial u\,\partial v} - \Gamma^u_{uv}\omega_u - \Gamma^v_{uv}\omega_v \tag{4.14}$$

Comparing this expression with (2.15), we obtain a simple result:

$$b_{uv} = \omega_{uv} \tag{4.15}$$

where ω_{uv} is the second covariant derivative of the phase ω. This result is valid not only for the indices (u, v), but also for other combinations of indices (u, u), and (v, v).

The formulae obtained enable us to find the values of B_u, B_v, C_u and C_v (see (4.21) and (4.22) below). The expressions for B_n and C_n contain the quantities

$$\beta_u = n_x\frac{\partial b_x}{\partial u} + n_y\frac{\partial b_y}{\partial u} + n_z\frac{\partial b_z}{\partial u} \tag{4.16}$$

$$\beta_v = n_x\frac{\partial b_x}{\partial v} + n_y\frac{\partial b_y}{\partial v} + n_z\frac{\partial b_z}{\partial v} \tag{4.17}$$

Let us calculate one of them. By virtue of $(\mathbf{b}\cdot\mathbf{n}) = 0$ we have

$$\beta_u = -\left(b_x\frac{\partial n_x}{\partial u} + b_y\frac{\partial n_y}{\partial u} + b_z\frac{\partial n_z}{\partial u}\right) \tag{4.18}$$

Inserting expressions (3.05) for b_x, b_y and b_z, and making use of (2.20), we obtain

$$\beta_u = G_{uu}\omega^u + G_{uv}\omega^v \tag{4.19}$$

Similarly

$$\beta_v = G_{vu}\omega^u + G_{vv}\omega^v \tag{4.20}$$

Now we can write out the new components of all the vectors. We have

$$\begin{aligned} B_u &= g_{uu} - \omega_u\omega_u + s(\omega_{uu} - \cos\vartheta\, G_{uu}),\\ B_v &= g_{uv} - \omega_u\omega_v + s(\omega_{uv} - \cos\vartheta\, G_{uv}),\\ B_n &= -\omega_u\cos\vartheta + s\left[G_{uu}\omega^u + G_{uv}\omega^v + \frac{\partial(\cos\vartheta)}{\partial u}\right]; \end{aligned} \tag{4.21}$$

$$\begin{aligned} C_u &= g_{vu} - \omega_v\omega_u + s(\omega_{vu} - \cos\vartheta\, G_{vu}),\\ C_v &= g_{vv} - \omega_v\omega_v + s(\omega_{vv} - \cos\vartheta\, G_{vv}),\\ C_n &= -\omega_v\cos\vartheta + s\left[G_{vu}\omega^u + G_{vv}\omega^v + \frac{\partial(\cos\vartheta)}{\partial v}\right]; \end{aligned} \tag{4.22}$$

We also have, according to (4.07)

$$A_u = \omega_u; \qquad A_v = \omega_v; \qquad A_n = \cos\vartheta \tag{4.07}$$

With these values of **A**, **B**, **C** the determinant, $D(s)$ which gives the cross-section of the bundle of rays, will be equal to

$$D(s) = \frac{1}{g}\begin{vmatrix} A_u & A_v & A_n \\ B_u & B_v & B_n \\ C_u & C_v & C_n \end{vmatrix} \tag{4.23}$$

This expression for the determinant can be considerably simplified with the aid of the relations

$$\begin{aligned} A_u\omega^u + A_v\omega^v + A_n\cos\vartheta &= 1 \\ B_u\omega^u + B_v\omega^v + B_n\cos\vartheta &= 0 \\ C_u\omega^u + C_v\omega^v + C_n\cos\vartheta &= 0 \end{aligned} \tag{4.24}$$

These relations can be easily verified. From (3.06) we have

$$\omega_u\omega^u + \omega_v\omega^v = \sin^2\vartheta = 1 - \cos^2\vartheta \tag{4.25}$$

Taking the covariant derivatives of this expression with respect to u and to v (they coincide with the usual derivatives), we obtain after dividing by 2

$$\begin{aligned} \omega_{uu}\omega^u + \omega_{uv}\omega^v &= -\cos\vartheta\,\frac{\partial(\cos\vartheta)}{\partial u} \\ \omega_{vu}\omega^u + \omega_{vv}\omega^v &= -\cos\vartheta\,\frac{\partial(\cos\vartheta)}{\partial v} \end{aligned} \tag{4.26}$$

Inserting into (4.24) the explicit expressions (4.07), (4.21) and (4.22) for the components of vectors **A**, **B** and **C**. and making use of (4.25) and (4.26), we see that relations (4.24) are verified. The geometrical meaning of these relations is obvious. They express the fact that **A** is a unit vector of the normal to the wave surface, while the vectors **B** and **C** are perpendicular to **A**.

Multiplying the third column in (4.23) by $\cos\vartheta$, and making use of (4.24), we obtain

$$D(s)\cos\vartheta = \frac{1}{g}\begin{vmatrix} A_u & A_v & 1 \\ B_u & B_v & 0 \\ C_u & C_v & 0 \end{vmatrix} = \frac{1}{g}\begin{vmatrix} B_u & B_v \\ C_u & C_v \end{vmatrix} \tag{4.27}$$

This expression acquires a more elegant form, if we introduce the

symmetrical tensor

$$T_{uv} = g_{uv} - \omega_u \omega_v + s(\omega_{uv} - \cos\vartheta\, G_{uv}) \tag{4.28}$$

According to (4.21) and (4.22) we have then

$$B_u = T_{uu}; \qquad B_v = T_{uv} \tag{4.29}$$

$$C_u = T_{vu}; \qquad C_v = T_{vv} \tag{4.30}$$

and the determinant (4.27) will take the form

$$D(s)\cos\vartheta = \frac{1}{g}\begin{vmatrix} T_{uu} & T_{uv} \\ T_{vu} & T_{vv} \end{vmatrix} \tag{4.31}$$

If we introduce mixed components of the tensor T_{uv} by the formulae

$$T_v^u = \sum_r g^{ur} T_{rv} \tag{4.32}$$

then instead of (4.31) we can write

$$D(s)\cos\vartheta = \begin{vmatrix} T_u^u & T_v^u \\ T_u^v & T_v^v \end{vmatrix} \tag{4.33}$$

or expanding the determinant

$$D(s)\cos\vartheta = T_u^u T_v^v - T_v^u T_u^v \tag{4.34}$$

Thus, the calculation of the determinant $D(s)$ is reduced to the calculation of the tensor T_{uv}, which presents no difficulties.

5. *Differential Geometry of the Wave Surface*

According to (3.10), equations

$$x = x_0 + (\chi - \omega)a_x^* \qquad \text{etc.} \tag{5.01}$$

represent, with constant χ, the parametric equations of the reflected wave surface. Every point on the reflected wave surface corresponds to a definite point on the surface of the reflecting body; the corresponding points lie on one and the same ray. To these two points there correspond the same values of parameters u, v. These parameters u, v and the phase χ can be interpreted as curvilinear coordinates in space.

The square of the distance between two infinitely close points will be of the form

$$\mathrm{d}l^2 = \sum_{u,v} g_{uv}^*\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\chi^2 \tag{5.02}$$

In this expression the products of the differentials $\mathrm{d}u\,\mathrm{d}\chi$ and $\mathrm{d}v\,\mathrm{d}\chi$ are

absent, while the square of the differential $d\chi$ enters with the coefficient unity.

The quadratic form

$$d\tau^2 = \sum_{u,v} g^*_{uv}\, du\, dv \tag{5.03}$$

represents the square of an element of arc on the wave surface.

We shall now find the coefficients of this quadratic form. Recalling expressions (4.06) for the vectors **B** and **C**, we can write

$$g^*_{uu} = \mathbf{B}^2; \qquad g^*_{uv} = \mathbf{B}\cdot\mathbf{C}; \qquad g^*_{vv} = \mathbf{C}^2 \tag{5.04}$$

In calculating the scalar product and the squares of the vectors **B** and **C**, we can make use of their components (4.21) and (4.22). We shall have, for example,

$$\mathbf{B}^2 = \sum_{u,v} g^{uv} B_u B_v + B_n^2 \tag{5.05}$$

Using (4.24) and introducing notations

$$\gamma^{uv} = g^{uv} + \frac{\omega^u \omega^v}{\cos^2\vartheta} \tag{5.06}$$

we may write

$$\mathbf{B}^2 = \sum_{u,v} \gamma^{uv} B_u B_v \tag{5.07}$$

Denoting the summation indices by p, q, and making use of (4.29), we obtain from (5.04):

$$g^*_{uu} = \sum_{p,q} \gamma^{pq} T_{up} T_{uq} \tag{5.08}$$

Similarly

$$g^*_{uv} = \sum_{p,q} \gamma^{pq} T_{up} T_{vq} \tag{5.09}$$

$$g^*_{vv} = \sum_{p,q} \gamma^{pq} T_{vp} T_{vq} \tag{5.10}$$

Thus, the coefficients of the quadratic form (5.03) are expressed directly in terms of the tensor T_{uv}. Let g^* be the determinant

$$g^* = g^*_{uu} g^*_{vv} - g^*_{uv} g^*_{vu} \tag{5.11}$$

(the discriminant of the quadratic form (5.03)). From equations (5.08) to (5.10) we have

$$g^* = \operatorname{Det} \gamma^{pq} (\operatorname{Det} T_{uv})^2 \tag{5.12}$$

whence

$$g^* = g\, D(s)^2 \tag{5.13}$$

The element dS^* of the surface of a reflected wave, corresponding to the element dS of the reflecting surface, is equal to

$$dS^* = \sqrt{g^*}\cdot du\, dv = D(s)\sqrt{g}\cdot du\, dv = D(s)\, dS \tag{5.14}$$

as it should be.

The quantities T_{uv} are linear, and the quantities g^*_{uv} are quadratic functions of s. For $s=0$, we have

$$g^*_{uv}(0) = T_{uv}(0) = g_{uv} - \omega_u \omega_v \tag{5.15}$$

We note that this tensor is inverse to the tensor γ^{uv}.

With an arbitrary s, we can write

$$T_{uv}(s) = T_{uv}(0) + sT'_{uv}(0) \tag{5.16}$$

where, according to (4.28)

$$T'_{uv}(0) = \omega_{uv} - \cos\vartheta\, G_{uv} \tag{5.17}$$

and also

$$g^*_{uv}(s) = T_{uv}(0) + 2sT'_{uv}(0) + s^2 \sum_{p,q} \gamma^{pq} T'_{up}(0) T'_{qv}(0) \tag{5.18}$$

We consider next the calculation of the second quadratic form which determines the radii of curvature of the wave surface. Its definition is similar to (2.20), only the vector $\mathbf{n}$ must be replaced by the vector $\mathbf{a}^*$ of the normal to the wave surface, and the quantities $\partial x_0/\partial v$ etc. — by the quantities $\partial x/\partial v$ etc. i.e. by the components of the vector $\mathbf{C}$ defined by (4.06). According to this definition we have

$$-G^*_{uv}(s) = \frac{\partial a^*_x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial a^*_y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial a^*_z}{\partial u}\frac{\partial z}{\partial v} \tag{5.19}$$

But this expression has already been found when calculating g^*_{uv}. Using (4.08), we can write

$$-sG^*_{uv}(s) = \left(B_x - \frac{\partial x_0}{\partial u} + \omega_u a^*_x\right) C_x + \dots \tag{5.20}$$

where the rows of dots denote the products of the corresponding y and z-components. Hence

$$-sG^*_{uv}(s) = \mathbf{B}\cdot\mathbf{C} - C_u = g^*_{uv}(s) - T_{uv}(s) \tag{5.21}$$

Thus, the coefficients of the first and second quadratic form are connected with the tensor $T_{uv}(s)$ by the relation

$$g^*_{uv}(s) + sG^*_{uv}(s) = T_{uv}(s) \tag{5.22}$$

Using this as well as (5.16)and (5.18) we can find an explicit expression for $G^*_{uv}(s)$ namely

$$-G^*_{uv}(s) = T'_{uv}(0)+s\sum_{p,q}\gamma^{pq}T'_{up}(0)T'_{qv}(0) \tag{5.23}$$

In particular, with $s=0$, we will have using (5.17)

$$-G^*_{uv}(0) = \omega_{uv}-\cos\vartheta\, G_{uv} \tag{5.24}$$

We have thus found the first as well as the second quadratic form for the reflected wave.

Similar calculations can be also carried out for the incident wave. For this, it is sufficient to replace in (3.07) and in the other formulae $\mathbf{a}^*$ by $\mathbf{a}$ (formula (3.01), and consider s as negative, so that $(-s)$ is the distance measured along the ray up to the point of incidence on the surface. We shall confine ourselves to formulae for the values of the coefficients $g^0_{uv}(0)$ and $G^0_{uv}(0)$ of the first and second quadratic form of the incident wave at the point of incidence of the ray. We have

$$g^0_{uv}(0) = g_{uv}-\omega_u\omega_v \tag{5.25}$$

$$-G^0_{uv}(0) = \omega_{uv}+\cos\vartheta\, G_{uv} \tag{5.26}$$

From these formulae, it is evident that the values of g_{uv} and of g^*_{uv} on the reflecting body coincide, while those of G^0_{uv} and of G^*_{uv} differ by the sign of the term containing $\cos\vartheta$. It is convenient to use relation (5.26) in the case when the incident wave is plane; then $G^0_{uv}=0$ and, consequently,

$$\omega_{uv} = -\cos\vartheta\, G_{uv} \tag{5.27}$$

Inserting this value in (4.28) we obtain

$$T_{uv} = g_{uv}-\omega_u\omega_v-2s\cos\vartheta\, G_{uv} \tag{5.28}$$

Calculating the quantity $D(s)$ from the formula (4.34) and using (4.25), we have after cancelling out $\cos\vartheta$,

$$D(s) = \cos\vartheta-2s\left(G\cos^2\vartheta+\sum_{u,v}G_{uv}\omega^u\omega^v\right)+4s^2\cos\vartheta\, K \tag{5.29}$$

Here K and G have the values (2.25) and (2.26). In order to show the geometrical meaning of the sum in the second term of (5.29), we note that if du and dv are the components of displacement on the surface of a reflecting body in the plane of incidence of the ray, and dσ is the magnitude of this displacement; then we have

$$\frac{du}{d\sigma} = \frac{\omega^u}{\sin\vartheta}\,; \qquad \frac{dv}{d\sigma} = \frac{\omega^v}{\sin\vartheta} \tag{5.30}$$

Therefore, designating by R_0 the radius of curvature of the section of the surface in the plane of incidence, we have

$$-\frac{1}{R_0} = \sum_{u,v} G_{uv} \frac{du}{d\sigma} \frac{dv}{d\sigma} = \frac{1}{\sin^2 \vartheta} \sum_{u,v} G_{uv} \omega^u \omega^v \tag{5.31}$$

Inserting the value of the sum into (5.29) and expressing G and K according to (2.25) and (2.26) by the principal radii of curvature we have for the case of a plane incident wave the following expression for $D(s)$:

$$D(s) = \cos \vartheta + 2s \left[\left(\frac{1}{R_1} + \frac{1}{R_2} \right) \cos^2 \vartheta + \frac{\sin^2 \vartheta}{R_0} \right] + \frac{4s^2}{R_1 R_2} \cos \vartheta \tag{5.32}$$

This formula was derived in Chapter 6.

6. *Reflection Formula*

The results obtained enable us to find (to the approximation of geometrical optics) the electromagnetic field of the reflected wave. We wrote the field of the incident wave in the form

$$\mathbf{E}^0 \, e^{ik\psi}; \qquad \mathbf{H}^0 \, e^{ik\psi} \tag{6.01}$$

As $\omega(u, v)$ is the value of the phase ψ on the surface of the reflecting body, then the field of the incident wave on this surface will be equal to

$$\mathbf{E}^0(u, v) \, e^{ik\omega}; \qquad \mathbf{H}^0(u, v) \, e^{ik\omega} \tag{6.02}$$

where $\mathbf{E}^0(u, v)$ and $\mathbf{H}^0(u, v)$ are the values of the amplitudes $\mathbf{E}^0$ and $\mathbf{H}^0$ on the surface of the body. Knowing $\mathbf{E}^0(u, v)$ and $\mathbf{H}^0(u, v)$ it is possible to find from the Fresnel formulae (given in Section 1) the values $\mathbf{E}^*(u, v)$ and $\mathbf{H}^*(u, v)$ of the field of the reflected wave on the surface of the body. The field of the reflected wave on this surface wili be equal to

$$\mathbf{E}^*(u, v) \, e^{ik\omega}; \qquad \mathbf{H}^*(u, v) \, e^{ik\omega} \tag{6.03}$$

Thus, the values (6.03) can be considered as known (at least on the illuminated part of the surface, sufficiently distant from the boundary of the shadow).

We want to find the field at a certain distance from the surface. For each of the components of the electromagnetic field this problem reduces to the following: it is necessary to find a function F satisfying the wave equation

$$\Delta F + k^2 F = 0 \tag{6.04}$$

and the radiation condition, and having on the surface of the body the

value

$$F = f(u, v)\, e^{ik\omega(u, v)} \tag{6.05}$$

In our case k is a large parameter, and $f(u, v)$ is a slowly varying function. The last assertion is to be understood in the sense that the derivatives, divided by k, of the function in directions tangential to the surface are small as compared to the values of the function itself. It is easy in this case to indicate an approximate solution of our problem. Obviously, the phase of the desired function will be obtained by replacing ω by

$$\chi = \omega + s \tag{6.06}$$

where s is the path the ray travelled after reflection. Its amplitude, however, will change in inverse proportion to the square root of the cross-section of the bundle of reflected rays. Thus, we arrive at the formula

$$F = f(u, v) \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik\chi} \tag{6.07}$$

where χ has the value (6.06).

Formula (6.07) can be derived in the following manner. Let us try to find F in the form

$$F = \sqrt{\varrho} \cdot e^{ik\chi'} \tag{6.08}$$

where ϱ and χ' are certain functions of the coordinates, not dependent upon the parameter k. Inserting (6.08) into the wave equation (6.04), we find

$$\Delta F + k^2 F = e^{ik\chi'} \left[k^2 \sqrt{\varrho} \cdot (1 - (\operatorname{grad} \chi')^2) + \frac{ik}{\sqrt{\varrho}} \operatorname{div} (\varrho \operatorname{grad} \chi') + \Delta (\sqrt{\varrho}) \right] \tag{6.09}$$

The wave equation will be approximately satisfied, if in expression (6.09) the terms of order k^2 and of the order k vanish. The phase χ' and the amplitude square ϱ must thus satisfy the equations

$$(\operatorname{grad} \chi')^2 = 1 \tag{6.10}$$

$$\operatorname{div} (\varrho \operatorname{grad} \chi') = 0 \tag{6.11}$$

Let us introduce now the curvilinear coordinates u, v, χ connected to the rectangular Cartesian coordinates x, y, z by means of relations (3.10), and write the equations (6.10) and (6.11) in these curvilinear coordinates. Introducing the tensor g^{*uv} according to formulae, similar to (2.06), inverse to the tensor g^*_{uv} defined by the formulae (5.08) to (5.10), we

have instead of (6.10)

$$\sum_{u,v} g^{*uv} \frac{\partial \chi'}{\partial u} \frac{\partial \chi'}{\partial v} + \left(\frac{\partial \chi'}{\partial \chi}\right)^2 = 1 \tag{6.12}$$

and instead of (6.11)

$$\frac{1}{\sqrt{g^*}} \left[\sum_{u,v} \frac{\partial}{\partial v} \left(\varrho \sqrt{g^*} \cdot g^{*uv} \frac{\partial \chi'}{\partial u} \right) + \frac{\partial}{\partial \chi} \left(\varrho \sqrt{g^*} \cdot \frac{\partial \chi'}{\partial \chi} \right) \right] = 0 \tag{6.13}$$

Equation (6.12) is satisfied, if we put

$$\chi' = \chi \tag{6.14}$$

Equation (6.13) is then reduced to the form

$$\frac{\partial}{\partial \chi} (\varrho \sqrt{g^*}) = 0 \tag{6.15}$$

and since according to (5.13)

$$\sqrt{g^*} = \sqrt{g} \cdot D(s) \tag{6.16}$$

where g does not depend on χ it will be satisfied if we put

$$\varrho D(s) = \varphi(u, v) \tag{6.17}$$

where φ is an arbitrary function of u, v.

In order to have agreement with (6.07), it is sufficient to assume that

$$\sqrt{\varrho} = f(u, v) \sqrt{\left(\frac{D(0)}{D(s)}\right)} \tag{6.18}$$

Thus, we have proved that function (6.07) approximately satisfies the wave equation (6.04). Obviously, it also satisfies the radiation condition (its phase increases with increasing s). Finally, for $s=0$, it reduces to the given function (6.05). Consequently, it satisfies all the requirements that were set up.

Applying expression (6.07) to the field of the reflected wave, and adding to it the field of the incident wave, we obtain the reflection formula in the form

$$\mathbf{E} = \mathbf{E}^0 \, e^{ik\varphi} + \mathbf{E}^*(u, v) \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik\chi} \tag{6.19}$$

$$\mathbf{H} = \mathbf{H}^0 \, e^{ik\varphi} + \mathbf{H}^*(u, v) \sqrt{\left(\frac{D(0)}{D(s)}\right)} \cdot e^{ik\chi} \tag{6.20}$$

In conclusion, we wish to mention that if the reflected body is convex, the reflection formula is applicable in the entire illuminated space sufficiently removed from the boundaries of the shadow (and at large distances from the body as well). If, however, the body is concave, then for certain values of s the denominator $D(s)$ can vanish (focal surfaces and lines). In the neighbourhood of the focal lines and surfaces, the geometric optics and, in particular, the reflection formula, is not applicable, since the condition that the amplitude be a slowly varying function of coordinates is not fulfilled there.

The transition of the reflection formula on the boundary of the shadow into the diffraction formulae has been investigated (for a plane incident wave and for small distances from the surface of the body) in Chapter 6.

7. *Reflection of a Spherical Wave from the Surface of a Sphere*

As an example of the application of the formulae obtained above let us examine the reflection of a spherical wave from the surface of a sphere. Let r, ϑ, φ, be spherical coordinates. The equation of the reflecting surface is of the form $r=a$. The part of the Gaussian parameters, u, v is played by the angles ϑ, φ, so that in our general formulae we may assume

$$u = \vartheta; \qquad v = \varphi \tag{7.01}$$

Let the source be situated at the point $\vartheta=0$, $r=b$. The surface value of the phase of the incident wave will then be

$$\omega(\vartheta, \varphi) = \sqrt{(a^2+b^2-2ab\cos\vartheta)} \tag{7.02}$$

The square of the line element on the sphere is

$$d\sigma^2 = a^2(d\vartheta^2+\sin^2\vartheta\, d\varphi^2) \tag{7.03}$$

whence

$$g_{\vartheta\vartheta} = a^2; \qquad g_{\vartheta\varphi} = 0; \qquad g_{\varphi\varphi} = a^2\sin^2\vartheta \tag{7.04}$$

$$\sqrt{g} = a^2 \sin\vartheta \tag{7.05}$$

$$g^{\vartheta\vartheta} = \frac{1}{a^2}; \qquad g^{\vartheta\varphi} = 0; \qquad g^{\varphi\varphi} = \frac{1}{a^2\sin^2\vartheta} \tag{7.06}$$

By the property of the sphere, the second differential form will be proportional to the first, and we have

$$G_{\vartheta\vartheta} = -a; \qquad G_{\vartheta\varphi} = 0; \qquad G_{\varphi\varphi} = -a\sin^2\vartheta \tag{7.07}$$

The covariant derivatives of the phase ω will be equal to

$$\omega_{\vartheta} = \frac{ab \sin \vartheta}{\omega}; \qquad \omega_{\varphi} = 0 \tag{7.08}$$

and the contra-variant derivatives will be

$$\omega^{\vartheta} = \frac{b \sin \vartheta}{a\omega}; \qquad \omega^{\varphi} = 0 \tag{7.09}$$

The angle of incidence of the ray (which we shall designate now by γ, since the letter ϑ has already been assigned) will be determined from the equations

$$\sin \gamma = \frac{b \sin \vartheta}{\omega}; \qquad \cos \gamma = \frac{b \cos \vartheta - a}{\omega} \tag{7.10}$$

following from (4.25). In order to calculate the second covariant derivatives of the phase, let us form according to formulae (2.08)–(2.11) the Christoffel symbols. We have

$$\begin{aligned} &\Gamma^{\vartheta}_{\vartheta\vartheta} = 0; \quad &&\Gamma^{\vartheta}_{\vartheta\varphi} = 0; \quad &&\Gamma^{\vartheta}_{\varphi\varphi} = -\sin \vartheta \cos \vartheta \\ &\Gamma^{\varphi}_{\vartheta\vartheta} = 0; \quad &&\Gamma^{\varphi}_{\vartheta\varphi} = \cot \vartheta; \quad &&\Gamma^{\varphi}_{\varphi\varphi} = 0 \end{aligned} \tag{7.11}$$

Inserting these values into the general formulae (2.15), we obtain

$$\begin{aligned} \omega_{\vartheta\vartheta} &= \frac{ab}{\omega^3}(b \cos \vartheta - a)(b - a \cos \vartheta) \\ \omega_{\vartheta\varphi} &= 0 \\ \omega_{\varphi\varphi} &= \frac{ab}{\omega} \sin^2 \vartheta \cos \vartheta \end{aligned} \tag{7.12}$$

Now we can form the tensor T_{uv}. We have

$$\begin{aligned} T_{\vartheta\vartheta} &= \frac{a^2}{\omega^2}(b \cos \vartheta - a)^2 + \frac{sa}{\omega^3}(b \cos \vartheta - a)(\omega^2 + b^2 - ab \cos \vartheta) \\ T_{\vartheta\varphi} &= 0 \\ T_{\varphi\varphi} &= a^2 \sin^2 \vartheta + \frac{sa}{\omega} \sin^2 \vartheta (2b \cos \vartheta - a) \end{aligned} \tag{7.13}$$

Let us now go over to the mixed components $T^{\vartheta}_{\vartheta}$ etc., and express $b \sin \vartheta$ and $b \cos \vartheta$ with the aid of (7.10) in terms of a, ω and γ. We obtain

$$\begin{aligned} T^{\vartheta}_{\vartheta} &= \frac{\cos \gamma}{\omega}\left[(s+\omega)\cos \gamma + \frac{2s\omega}{a}\right] \\ T^{\varphi}_{\varphi} &= \frac{1}{\omega}\left(s + \omega + \frac{2s\omega}{a}\cos \gamma\right) \end{aligned} \tag{7.14}$$

whereas

$$T_{\varphi}^{\vartheta} = T_{\vartheta}^{\varphi} = 0$$

By virtue of (4.34) the quantity $D(s) \cos \gamma$ is equal to the product of the quantities (7.14). Hence

$$\omega^2 D(s) = \left[(s+\omega) \cos \gamma + \frac{2s\omega}{a}\right]\left(s+\omega+\frac{2s\omega}{a} \cos \gamma\right) \tag{7.15}$$

This expression is symmetrical with respect to s and ω.

Our results enable us at once to write out the reflection formula for the vertical component of the electric and magnetic Hertzian vector, which satisfies the scalar wave equation.

Designating the distance from the source with the letter R where

$$R = \sqrt{(b^2+r^2-2br \cos \vartheta)} \tag{7.16}$$

we have for the Hertzian electric vector

$$U = \frac{e^{ikR}}{R} + N\frac{e^{ik\omega}}{\omega} \sqrt{\left(\frac{D(0)}{D(s)}\right)} e^{iks} \tag{7.17}$$

where N is the Fresnel coefficient (1.10). For the Hertzian magnetic vector, the formula will be the same, only instead of N there will be the other Fresnel coefficient M.

Introducing the expression (7.15) for $D(s)$, and putting for the sake of brevity

$$\frac{2s\omega}{a(s+\omega)} = c_1 \tag{7.18}$$

we have

$$U = \frac{e^{ikR}}{R} + \frac{N}{\omega+s} \sqrt{\left(\frac{\cos \gamma}{(\cos\gamma + c_1)(1+c_1\cos\gamma)}\right)} e^{ik(\omega+s)} \tag{7.19}$$

This formula can be compared with that obtained from the diffraction formulae derived in Chapter 12 for the case of a glancing ray incidence, and for distances from the surface of the sphere which are small as compared with its radius. The formula indicated reduces to the form

$$U = \frac{e^{ikR}}{R}\left(1+\frac{p+iq}{p-iq} \sqrt{\left(\frac{p}{p+p_1}\right)} e^{2ip_1p^2}\right) \tag{7.20}$$

Here

$$p = m \cos \gamma; \qquad p_1 = mc_1; \qquad q = \frac{im\sqrt{(\eta-1)}}{\eta} \tag{7.21}$$

and

$$m = \sqrt[3]{\left(\frac{ka}{2}\right)}\ldots \tag{7.22}$$

A necessary condition for the applicability of the reflection formula (7.20) is that the quantity p should be large and positive; if, however, p is of the order of unity, then the diffraction formulae must be used.

It is not difficult to see that formula (7.20), with certain approximations, coincides with (7.19). Since the values c_1 and $\cos\gamma$ are small relative to unity, their product in (7.19) can be neglected. Further, the quantity $\omega+s$ in the denominator can be replaced by R. For the same quantity in the exponential function, we can use the expression

$$\omega+s-R = \frac{4\omega s\cos^2\gamma}{\omega+s+R} \tag{7.23}$$

whence approximately

$$k(\omega+s-R) = kac_1\cos^2\gamma = 2p_1p^2 \tag{7.24}$$

Further,

$$\frac{\cos\gamma}{\cos\gamma+c_1} = \frac{p}{p+p_1} \tag{7.25}$$

Finally, we have for small $\cos\gamma$ and for $\mu=1$

$$N = \frac{p+iq}{p-iq} \tag{7.26}$$

If we use these approximate expressions, the agreement between (7.19) and (7.20) will be complete.

CHAPTER 9

ON THE TRANSVERSE DIFFUSION OF SHORT WAVES DIFFRACTED BY A CONVEX CYLINDER†

Abstract — The two-dimensional problem of diffraction of short waves on a convex cylinder of arbitrary section is reduced to the solution of a parabolic equation expressed in ray coordinates; this equation takes into account the transverse diffusion but neglects the longitudinal diffusion. The radius of curvature of the cylinder section is supposed to be large (as compared with the wavelength) and slowly varying. The boundary conditions considered are of impedance type (with a special dependence of the impedance parameter on the curvature radius); the most important case is that of a perfectly reflecting cylinder. Under the assumptions stated above an asymptotic solution of the diffraction problem is obtained for an arbitrary position of the source and of the observation point.

1. *Introduction*

It is well known that the field near the surface of a conducting convex body (with boundary conditions of impedance type) is expressible in terms of universal functions first introduced by Fock [5]. The original expressions were given for the penumbra region, but the investigation of the diffraction on a sphere and on a circular and an elliptic cylinder [30], [31], [32] has shown that the same universal functions are valid in the deep shadow. This suggests the possibility of finding generalized expressions describing the diffraction on other surfaces with variable curvature and valid for any distances from the surface.

Such generalizations can be based, on the one hand, on the conception of diffracted rays [26] and, on the other hand, on the conception of transverse diffusion [29]; the latter constitutes a physical interpretation of Leontovich's parabolic equation.

2. *Ray Coordinates*

In the present communication we consider a two-dimensional diffraction problem, namely the diffraction of a cylindrical wave on a convex cylinder of arbitrary section. We use "ray coordinates" introduced by Malyughi-

† Fock and Wainstein, 1962.

netz [31], [32] and defined according to Fig. 1 and 2. Let ξ, η be ray coordinates for the observation point P and ξ', η' those for the (two-dimensional) source P'. The straight lines PT and $P'T'$ are tangents to the cylinder. The positions of the tangent points T and T' are characterized by means of the two arcs

$$\eta = T_0T \qquad \text{and} \qquad \eta' = T_0T' \tag{2.01}$$

where the point T_0 on the contour (the origin of the arc) is arbitrary. The positions of the points P and P' are characterized by the two distances

$$\xi - \eta = PT \qquad \text{and} \qquad \eta' - \xi' = P'T' \tag{2.02}$$

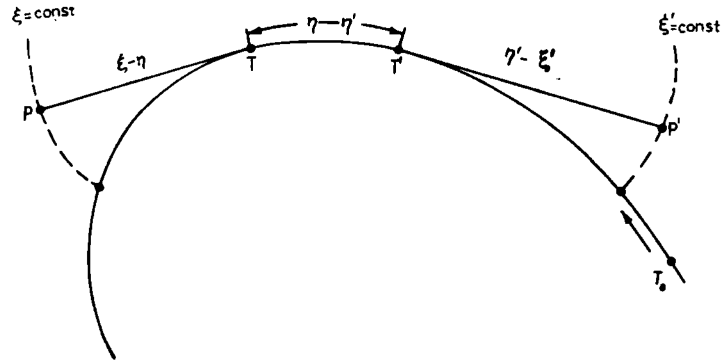

FIG. 1.

We choose the positive direction of the arcs so that $\xi > \xi'$; then if P is in the shadow region (Fig. 1) we have $\eta > \eta'$ and the total length $P'P$ (including two straight parts and a curved part $\eta - \eta'$ on the cylinder surface) will be equal to

$$(\eta' - \xi') + (\eta - \eta') + (\xi - \eta) = \xi - \xi' \tag{2.03}$$

The expression for the square of the line-element in the plane of the normal section of the cylinder in terms of the ray coordinates is of the form

$$ds^2 = d\xi^2 + \frac{(\xi - \eta)^2}{\varrho^2(\eta)}\, d\eta^2 \tag{2.04}$$

where $\varrho(\eta)$ is the radius of curvature of the cylinder. The quantity

$$d\vartheta = \frac{d\eta}{\varrho(\eta)} \tag{2.05}$$

is the increment of the angle ϑ between the tangent and a fixed direction. Since the equation of the cylinder is $\xi-\eta = 0$, the tangent line-element is $ds=d\xi$ and the line-element along the normal near the surface is

$$dn = (\xi-\eta)\, d\vartheta = \frac{\xi-\eta}{\varrho(\eta)}\, d\eta \qquad (2.06)$$

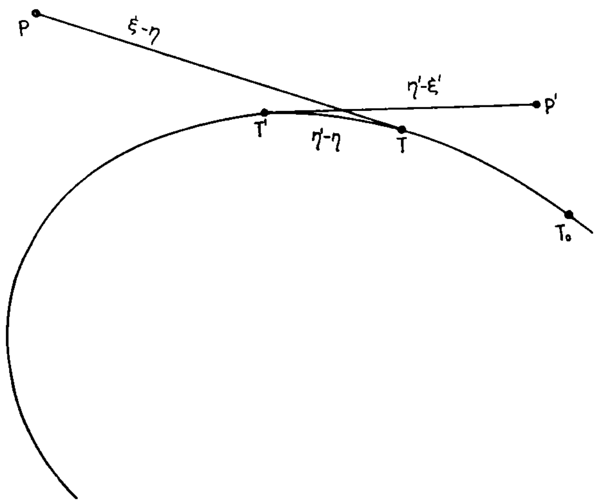

FIG. 2.

Introducing the angle ϑ defined above we can form the expressions

$$X = \xi \cos \vartheta + \int_0^{\vartheta} \eta \sin \vartheta \, d\vartheta$$
$$Y = \xi \sin \vartheta - \int_0^{\vartheta} \eta \cos \vartheta \, d\vartheta \qquad (2.07)$$

which can be interpreted as rectangular coordinates in the plane of the cylinder section.

The ray coordinates defined above can also be used in that part of the penumbra region where $\eta < \eta'$ (see Fig. 2).

3. *Differential Equations*

We introduce the Green's function Γ in the multi-layered Riemann plane. This is a function of two pairs of coordinates: the coordinates ξ, η of the observation point and the coordinates ξ', η' of the source. The

Green's function Γ must satisfy the time-independent wave equation

$$\Delta\Gamma+k^2\Gamma = 0 \tag{3.01}$$

with the boundary condition

$$\frac{\partial\Gamma}{\partial n}+ikg\Gamma = 0 \tag{3.02}$$

on the cylinder surface and must fulfil the radiation condition at infinity. We take the incident cylindrical wave in the form

$$\Gamma^0 = i\pi H_0^{(1)}(kr) \tag{3.03}$$

r being the distance between P and P'. Then the singularity of Γ must be such that the difference $\Gamma-\Gamma^0$ as well as its derivatives is everywhere finite and continuous.

For a cylinder of infinite perimeter the function Γ reduces to the ordinary Green's function; in the general case the latter is expressible in terms of Γ.

It is easily seen from (2.04) that the wave equation in ray coordinates assumes the form

$$\frac{1}{\xi-\eta}\frac{\partial}{\partial\xi}\left[(\xi-\eta)\frac{\partial\Gamma}{\partial\xi}\right]+\frac{\varrho(\eta)}{\xi-\eta}\frac{\partial}{\partial\eta}\left[\frac{\varrho(\eta)}{\xi-\eta}\frac{\partial\Gamma}{\partial\eta}\right]+k^2\Gamma = 0 \tag{3.04}$$

It follows that the substitution

$$\Gamma = e^{ik(\xi-\xi')}W \tag{3.05}$$

reduces the differential equation (3.04) to the form

$$\left[\frac{1}{\xi-\eta}\frac{\partial}{\partial\xi}(\xi-\eta)\frac{\partial}{\partial\xi}+D^2+ikR\right]W = 0 \tag{3.06}$$

where D and R denote the operators

$$D = \frac{\varrho(\eta)}{\xi-\eta}\frac{\partial}{\partial\eta}; \qquad R = 2\frac{\partial}{\partial\xi}+\frac{1}{\xi-\eta} \tag{3.07}$$

The operator D (as well as D_1, D_2 and D_3 introduced later) is associated with transverse diffusion and may be called the "diffusion operator". The operator R (as well as $R_\gamma R_\beta$ and R_α, see later) is associated with the law of decrease of the amplitude of a cylindrical wave and may be called the "ray amplitude operator" or, simpler the "ray operator". The second order differential operator in the first term of (3.06) is connected with longitudinal diffusion.

If both diffusion operators are neglected, equation (3.06) assumes the form

$$RW = 0 \tag{3.08}$$

and has a solution

$$W = \frac{W(\eta)}{\sqrt{(\xi-\eta)}} \tag{3.09}$$

showing the law of decrease of amplitude in a region where diffusion is absent (region GD, see Fig. 3 in Section 6 below).

In order to estimate the relative importance of the two diffusino operators, we introduce the dimensionless quantity

$$M(\eta) = \left[\frac{k\varrho(\eta)}{2}\right]^{1/3} \tag{3.10}$$

and make a transformation from ξ, η to new independent variables

$$z = \frac{k}{2}\int_0^\eta \frac{\mathrm{d}\bar{\eta}}{M^2(\bar{\eta})}, \qquad \gamma = \frac{k(\xi-\eta)}{2M^2(\eta)} \tag{3.11}$$

Taking as the new dependent variable the function W_1 defined by

$$W = \frac{1}{\sqrt{M(\eta)}} W_1(z, \gamma) \tag{3.12}$$

we obtain for W_1 the equation

$$\left[L^2+D_1^2+\left(\chi+\frac{1}{\gamma}\right)D_1+2iR_\gamma\right]W_1 = 0 \tag{3.13}$$

where L, D_1 and R_γ are the operators

$$L = \frac{1}{M(\eta)}\cdot\gamma\frac{\partial}{\partial\gamma}; \qquad D_1 = \frac{\partial}{\partial z}-(1+2\chi\gamma)\frac{\partial}{\partial\gamma}-\frac{\chi}{2} \tag{3.14}$$

$$R_\gamma = 2\gamma^2\frac{\partial}{\partial\gamma}+\gamma \tag{3.15}$$

and χ denotes the function

$$\chi(z) = \frac{\mathrm{d}\log M(\eta)}{\mathrm{d}z} = \frac{2M(\eta)}{k}\,\frac{\mathrm{d}M(\eta)}{\mathrm{d}\eta} \tag{3.16}$$

The transformation from (3.06) to (3.13) is most easily performed with the help of the relations

$$\left.\begin{aligned} \frac{1}{\xi-\eta}\frac{\partial}{\partial\xi}(\xi-\eta)\frac{\partial}{\partial\xi} &= \frac{k}{4M^2\gamma^2}L^2 \\ \sqrt{M}\cdot D^2\frac{1}{\sqrt{M}} &= \frac{k^2}{4M^2\gamma^2}\left[D_1^2+\left(\chi+\frac{1}{\gamma}\right)D_1\right] \\ \sqrt{M}\cdot kR\frac{1}{\sqrt{M}} &= \frac{k^2}{4M^2\gamma^2}\cdot 2R_\gamma \end{aligned}\right\} \tag{3.17}$$

connecting the respective operators. The operator L corresponds to longitudinal and D_1 to transverse diffusion and R_γ is the new ray operator.

Now, if the radius of curvature $\varrho(\eta)$ along the arc $T'T$ is large as compared with the wavelength, the quantity M defined by (3.10) can be considered as large. Since the operator L involves M in the denominator, the L^2 term in (3.13) is small of the order $1/M^2$ as compared with the other terms. Neglecting it (and thus neglecting longitudinal diffusion) we obtain the parabolic equation

$$\left[D_1^2+\left(\chi+\frac{1}{\gamma}\right)D_1+2i\,R_\gamma\right]W_1 = 0 \tag{3.18}$$

which corresponds to the equation

$$(D^2+ikR)W = 0 \tag{3.19}$$

in the original variables ξ, η.

A further transformation of variables is suitable for some purposes, especially for the study of the solution near the surface of the cylinder. We put

$$W_1 = \frac{1}{\sqrt{(1+\chi\gamma)}}W_2(z,\beta), \qquad \beta = \frac{\gamma}{1+\chi\gamma} \tag{3.20}$$

and introduce the notation

$$\left.\begin{aligned} D_2 &= \frac{\partial}{\partial z}-\frac{\partial}{\partial\beta}+2\varkappa R_\beta \\ R_\beta &= 2\beta^2\frac{\partial}{\partial\beta}+\beta \end{aligned}\right\} \tag{3.21}$$

where

$$\varkappa = \varkappa(z) = \frac{1}{4}\left(\chi^2-\frac{d\chi}{dz}\right) = -\frac{M^3(\eta)}{k^2}\frac{d^2M(\eta)}{d\eta^2} \tag{3.22}$$

Then we have

$$\left.\begin{aligned}(1+\chi\gamma)^{\frac{1}{2}} D_1 (1+\chi\gamma)^{-\frac{1}{2}} &= D_2 \\ (1+\chi\gamma)^{\frac{1}{2}} R_\gamma (1+\chi\gamma)^{-\frac{1}{2}} &= R_\beta\end{aligned}\right\} \tag{3.23}$$

and equation (3.18) transforms into

$$\left(D_2^2+\frac{1}{\beta}D_2+2iR_\beta\right)W_2 = 0 \tag{3.24}$$

The boundary of the cylinder is given by the relation $\xi=\eta$ in the original coordinates and by the relations $\gamma=0$ and $\beta=0$ in the transformed coordinates.

By means of the substitution

$$\eta_2 = \int_0^\eta \frac{(\xi-\bar{\eta})}{\varrho(\bar{\eta})}\,\mathrm{d}\bar{\eta} \tag{3.25}$$

and a suitable change of the dependent variable it is also possible to reduce the parabolic equation (3.19) to the simple form

$$\frac{\partial^2 F}{\partial \eta_2^2}+2ik\frac{\partial F}{\partial \xi} = 0 \tag{3.26}$$

which is useful in some respects, e.g. for the investigation of the phase of the incident wave and of the Fresnel diffraction. But it is difficult to apply equation (3.26) to the boundary value problem, since the equation of the boundary expressed in the variables ξ, η_2 is in general complicated.

4. *Circular Cylinder*

As a preliminary stage we consider the solution of the equations stated above for the case of a circular cylinder. The quantity M is then a constant so that $\chi=0$ and $\varkappa=0$. The transformation (3.20) becomes an identity and we have $\gamma=\beta$. Equation (3.18) or (3.24) takes the form

$$\left[\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \beta}\right)^2+\frac{1}{\beta}\left(\frac{\partial}{\partial z}-\frac{\partial}{\partial \beta}\right)+2iR_\beta\right]W_2 = 0 \tag{4.01}$$

By means of the substitution

$$W_2 = \mathrm{e}^{-i\frac{2}{3}\beta^3}\,W_0(x, y); \qquad x = z+\beta, \quad y = \beta^2 \tag{4.02}$$

equation (4.01) reduces to the well-known form

$$\frac{\partial^2 W_0}{\partial y^2}+i\frac{\partial W_0}{\partial x}+yW_0 = 0 \tag{4.03}$$

The boundary condition (3.02) becomes

$$\frac{\partial \Gamma}{\partial y}+igM\Gamma = 0 \qquad \text{for} \qquad y = 0 \tag{4.04}$$

since we have, using (2.06),

$$k\,\mathrm{d}n = M\,\mathrm{d}y \tag{4.05}$$

The same condition (4.04) holds for the function W in (3.05) and also for W_2 and W_0. Putting

$$q = igM \tag{4.06}$$

we may write this condition in the form

$$\frac{\partial W}{\partial y}+qW = 0 \qquad \text{for} \qquad y = 0 \tag{4.07}$$

Near the singularity of the Green's function the parabolic equations do not hold and the requirement that $\Gamma-\Gamma^0$ should be finite at the singularity must be replaced by another one. We must impose the condition that on the boundary of geometrical shadow (far from the singularity) the incident wave should be the same as that given by Γ^0.

The solution satisfying these conditions can be expressed in terms of the attenuation factor $\Psi(x, y, y', q)$ defined (for $x>0$ and $y'>y$) by the integral

$$\Psi(x, y, y'\,q) = \frac{1}{2\pi i}\int_C e^{itx}\,w(t-y')\left[v(t-y)-\frac{v'(t)-qv(t)}{w'(t)-qw(t)}\,w(t-y)\right]\mathrm{d}t, \tag{4.08}$$

where $w(t)$ is the Airy function satisfying the differential equation

$$w''(t) = tw(t) \tag{4.09}$$

and having for $y\to\infty$ the asymptotic expression

$$w(t-y) = \frac{1}{y^{1/4}}\exp\left[i\left(\frac{2}{3}y^{3/2}-t\sqrt{y}+\frac{t^2}{4\sqrt{y}}+\frac{\pi}{4}\right)\right] \tag{4.10}$$

The Airy function $v(t)$ satisfies the same equation and is the imaginary part of $w(t)$ for real t. The contour C in (4.08) encloses the first quadrant

of the complex t-plane, where the poles of the integrand are situated, and runs in the positive direction.

The solution for a circular cylinder can be written as

$$W = \frac{2\pi i}{M} e^{-i\frac{2}{3}\left(y^{3/2}+y'^{3/2}\right)} \Psi(x-x', y, y', q) \tag{4.11}$$

The variables y and y' and their square roots $\sqrt{y}$ and $\sqrt{y'}$ are defined in accordance with (5.04) and (5.05) below.

Near the boundary of geometrical shadow the Ψ-function can be described as a Fresnel integral superimposed on a slowly varing function representing a weak background (Chapter 7). Putting in accordance with (3.11) and (4.02)

$$z = x-\sqrt{y}; \qquad z' = x'+\sqrt{y'}; \qquad z-z' = \zeta \tag{4.12}$$

and introducing the positive quantity μ defined by

$$\mu^2 = \frac{\sqrt{(yy')}}{\sqrt{y}+\sqrt{y'}} \tag{4.13}$$

we have for the Fresnel integral term

$$\Psi = \frac{1}{2\pi i} \frac{1}{\sqrt[4]{(yy')}} e^{i\frac{2}{3}\left(y^{3/2}+y'^{3/2}\right)} \mu F(\mu\zeta) \tag{4.14}$$

and consequently

$$W = \frac{1}{M\sqrt[4]{(yy')}} \mu F(\mu\zeta) \tag{4.15}$$

where

$$F(\tau) = e^{-i\tau^2} \int_{\tau}^{\infty} e^{it^2}\, dt \tag{4.16}$$

If μ is large, the terms neglected in (4.14) and (4.15) (the background) are small of the order $1/\mu$ with respect to the main term. (See also Ref. 27).

The expression for the incident wave follows from (4.15) for large negative values of $\mu\zeta$ and is approximately equal to

$$W^0 = \frac{\mu}{M\sqrt[4]{(yy')}} \sqrt{\pi}\, e^{i\frac{\pi}{4}} e^{-i\mu^2\zeta^2} \tag{4.17}$$

It is easy to verify that for small values of ζ the quantity $\mu^2\zeta^2$ is nearly equal to the phase difference $k(\xi-\xi'-r)$. Thus the expression (4.11) satisfies the differential equation, the boundary condition and the condition on the shadow boundary. It is therefore the required solution.

5. *Cylinder with a Variable Radius of Curvature and with $\varkappa=0$*

Equation (4.01) holds rigorously not only in the case $\chi=0$, but also in the case $\chi\neq 0$, $\varkappa=0$. The substitution resulting from (3.12) and (3.20) may be written as

$$W = \frac{1}{\sqrt{[N(\xi, \eta)]}} W_2 \tag{5.01}$$

where

$$N(\xi, \eta) = M(\eta)+(\xi-\eta)\frac{\mathrm{d}M(\eta)}{\mathrm{d}\eta} \tag{5.02}$$

is a linear function of ξ. If $\varkappa=0$, we have

$$N(\xi, \eta) = M(\xi)$$

We use the variables z and z' defined by (3.11) so that

$$z-z' = \zeta = \frac{k}{2}\int_{\eta'}^{\eta} \frac{\mathrm{d}\bar{\eta}}{M^2(\bar{\eta})} \tag{5.03}$$

and also the variables β and β' where

$$\begin{aligned} \beta &= \frac{k(\xi-\eta)}{2M(\eta)N(\xi, \eta)} = \frac{\gamma}{1+\chi(z)\gamma}; \qquad & \gamma &= \frac{k(\xi-\eta)}{2M^2(\eta)} \\ \beta' &= \frac{k(\xi'-\eta')}{2M(\eta)'N(\xi', \eta')} = \frac{\gamma'}{1+\chi(z')\gamma'}; \qquad & \gamma' &= \frac{k(\xi'-\eta')}{2M^2(\eta')} \end{aligned} \tag{5.04}$$

Since we are considering the case when $\xi-\eta>0$, $\xi'-\eta'<0$ we will have $\gamma>0$, $\gamma'<0$ and for moderate values of γ and γ' also $\beta>0$, $\beta'<0$. We take the square roots $\sqrt{y}$ and $\sqrt{y'}$ in (4.11) always positive; therefore, we have

$$\sqrt{y} = \beta; \qquad \sqrt{y'} = -\beta' \tag{5.05}$$

Introducing these values in (4.11) and putting for brevity

$$N = N(\xi, \eta); \qquad N' = N(\xi', \eta') \tag{5.06}$$

we can write as a generalization of the solution (4.11) for the case $\chi\neq 0$, $\varkappa=0$

$$W = \frac{2\pi i}{\sqrt{(NN')}}\mathrm{e}^{-i\frac{2}{3}(\beta^3-\beta'^3)}\Psi(\zeta+\beta-\beta', \beta^2, \beta'^2, q) \tag{5.07}$$

In the region where $\zeta=z-z'$ is small but β and $|\beta'| = -\beta'$ are large the function W can be approximately expressed in terms of Fresnel inte-

grals, as in the case of a circular cylinder. We will have

$$W = \frac{1}{\sqrt{(NN')}} \frac{1}{\sqrt[4]{(yy')}} \mu F(\mu\zeta) \tag{5.08}$$

where μ is defined by

$$\frac{1}{\mu^2} = \frac{1}{\sqrt{y}} + \frac{1}{\sqrt{y'}} = \frac{1}{\beta} - \frac{1}{\beta'} = \frac{1}{\gamma} - \frac{1}{\gamma'} + \chi(z) - \chi(z') \tag{5.09}$$

Since we are considering small values of ζ we can neglect the difference $\chi(z)-\chi(z')$ and also the difference between $M^2(\eta)$ and $M^2(\eta')$ in the definition (5.04) of γ and γ'. We obtain approximately

$$\frac{1}{\mu^2} = \frac{1}{\gamma} - \frac{1}{\gamma'} = \frac{2}{k} M(\eta)M(\eta') \frac{\xi-\xi'}{(\xi-\eta)(\eta'-\xi')} \tag{5.10}$$

With this value of μ and the definition (5.03) of ζ it is easy to see that the square of the argument in the Fresnel integral is equal to

$$\mu^2\zeta^2 = ks = k(\xi-\xi'-r) \tag{5.11}$$

This gives the correct value of the phase of the incident wave. On the other hand, the relation

$$NN'\sqrt{(yy')} = M(\eta)M(\eta')(-\gamma\gamma') = \frac{k^2}{4} \frac{(\xi-\eta)(\eta'-\xi')}{M(\eta)M(\eta')} \tag{5.12}$$

together with (5.10) shows that formula (5.08) may be written as

$$W = \sqrt{\left[\frac{2}{k(\xi-\xi')}\right]} F(\mu\zeta) \tag{5.13}$$

Using the expression for the Fresnel integral valid for large negative values of the argument we obtain for the incident wave

$$W^0 = \sqrt{\left[\frac{2\pi}{k(\xi-\xi')}\right]} e^{-iks+i\frac{\pi}{4}} \tag{5.14}$$

as it ought to be. It is possible to find for s an explicit expression such that W^0 is an exact solution of the parabolic equation (3.19) for the general case (not only for $\varkappa=0$).

We thus have shown that our solution (5.07) correctly represents the incident wave in a certain domain near the boundary between light and shadow.

The following remarks are appropriate at this point.

Firstly, expression (5.07) satisfies the boundary condition (4.07) with constant q. Since q is related to g and M by means of (4.06) the constancy of q is a rather artificial requirement. For a perfect conductor, however, we have for the value of g in the original boundary condition either $g=\infty$ or $g=0$ and these values give $q=\infty$ and $q=0$ respectively, so that the solution (5.07) is then applicable.

Secondly, it is possible that $N(\xi, \eta)$ vanishes for finite values of ξ, so that β becomes infinite for finite distances from the cylinder. The corresponding curve in the ξ, η-plane may be called diffractional caustic. There is no physical singularity near the diffractional caustic and if one uses the asymptotic expression (4.10) for the Airy function, one can verify that W passes smoothly across the diffractional caustic. Beyond the caustic the Airy function $w=w_1=u+iv$ is to be replaced by $w_2=u-iv$.

6. *The General Case of a Convex Cylinder*

The solution of the parabolic equation obtained in the preceding section is a rigorous solution only in the case $\varkappa=0$, but the formulae have been written so as to retain their meaning in the general case $\varkappa\neq 0$ as well. It is therefore plausible that these formulae yield an approximate solution also if $\varkappa$ is different from zero but small.

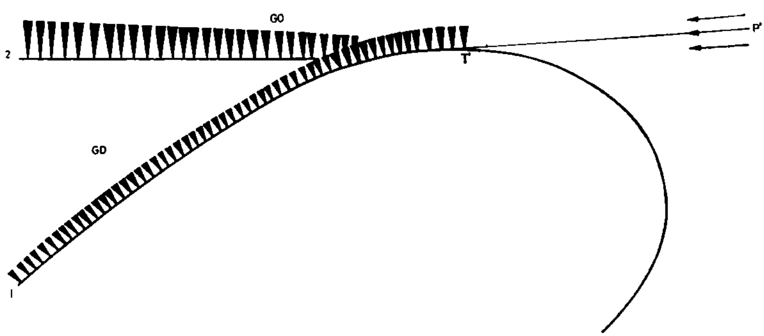

FIG. 3.

Before investigating this fundamental question we shall examine the physical aspect of the problem [29]. Suppose a plane or a cylindrical wave from a remote source is falling on a convex cylinder (Fig. 3). Then in the illuminated region GO the laws of geometrical optics apply. But there are two other regions (or zones) where the transverse diffusion plays a dominant part. One of them (zone 1) is close to the shadowed part of the cylin-

der surface and the other (zone 2) lies near the geometrical boundary between light and shadow, where the Fresnel diffraction takes place. Between the zones 1 and 2 a region *GD* is situated, where the notion of diffraction applies. In the regions *GO* and *GD* transverse diffusion is insignificant.

If we take the solution in the form (5.07) then in the first zone β is small or finite. In the second zone β is large, $\zeta = z - z'$ is small and $\mu\zeta$ is small or finite.

Let us suppose $\varkappa$ to be very small and let us see whether the transition from the full parabolic equation (3.24) to the abridged form (4.01) is legitimate in the first zone. This transition amounts to the neglect of the term $2\varkappa R_\beta$ in the operator $D_2 = \dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \beta} + 2\varkappa R_\beta$. Now, if $\varkappa$ and β are small the term $2\varkappa R_\beta W_2$ is obviously small as compared with other terms in $D_2 W_2$; this term can be neglected even in the expression $(1/\beta)\, D_2 W_2$ for $\beta \to 0$. Because of the smallness of $\varkappa$ the term $2\varkappa R_\beta W_2$ is also small for finite values of β. Thus equation (4.01) is a good substitute for (3.24) everywhere in the first zone, including the immediate vicinity of the cylinder surface. (It is to be noted that this is not the case for the equation obtained from (3.18) by neglecting the terms in χ).

Next, let us consider the *GD* region and the second zone where β is large. In the *GD* region the term $R_\beta W_2$ is of the same order as $\partial W_2/\partial z$; this follows from the fact that W_2 is approximately proportional to $1/\sqrt{\beta}$ (the factor appearing in (4.10)). Thus in the *GD* region the term $2\varkappa R_\beta$ is unimportant and can be neglected.

In the second zone, however, where the Fresnel diffraction takes place, the term $R_\beta W_2$ is of the order $\mu \cdot \partial W_2/\partial z$ so that $2\varkappa R_\beta W_2$ can be neglected only if $4\varkappa\mu \ll 1$ which is the case only in a part of the second zone. Therefore, the second zone must be investigated separately.

The simplest way is to go back to equation (3.18) and to neglect the terms in χ. Then equation (3.18) takes the form (4.01) with β replaced by γ. The solution can be written thus:

$$W = \frac{2\pi i}{\sqrt{[M(\eta)M(\eta')]}}\, e^{-i\frac{2}{3}(\gamma^3 - \gamma'^3)}\, \Psi(\zeta + \gamma - \gamma', \gamma^2, \gamma'^2, q) \tag{6.01}$$

The asymptotic form of this solution in the Fresnel zone is according to (5.08)

$$W = \frac{1}{\sqrt{[M(\eta)M(\eta')]}}\, \frac{1}{\sqrt{(-\gamma\gamma')}}\, \mu F(\mu\zeta) \tag{6.02}$$

with μ defined in terms of γ and γ' according to (5.10). Using (6.02) one

can verify that the terms in χ in the operator D_1 are small if χ is small so that the approximation (6.01) is self-consistent.

It is, however, possible to improve the approximation in the second zone by using in place of γ a slightly modified independent variable, namely

$$\alpha = \gamma \cdot \frac{1+a\eta}{1+a\xi} = \frac{\gamma}{1+\chi^0\gamma} \tag{6.03}$$

where a is a constant and

$$\chi^0 = \chi^0(z) = \frac{2M^2(\eta)}{k} \frac{a}{1+a\eta} = \frac{\mathrm{d}}{\mathrm{d}z} \log(1+a\eta) \tag{6.04}$$

This transformation corresponds to the change of the dependent variables

$$W_1 = \frac{1}{\sqrt{(1+\chi^0\gamma)}} W_3(z, \alpha) \tag{6.05}$$

The transformed operators are

$$(1+\chi^0\gamma)^{1/2} D_1 (1+\chi^0\gamma)^{-1/2} = D_3 \tag{6.06}$$

$$(1+\chi^0\gamma)^{1/2} R_\gamma (1+\chi^0\gamma)^{-1/2} = R_\alpha \tag{6.07}$$

where

$$D_3 = \frac{\partial}{\partial z} - \frac{\partial}{\partial \alpha} + (\chi^0 - \chi)\left(2\alpha \frac{\partial}{\partial \alpha} + \frac{1}{2}\right) \tag{6.08}$$

and

$$R_\alpha = 2\alpha^2 \frac{\partial}{\partial \alpha} + \alpha \tag{6.09}$$

We see that the transformation (6.03) does not introduce "long-range" terms of the type $2\varkappa R_\alpha$ in the operator D_3 and at the same time replaces the coefficient $(-\chi)$ in D_1, by the coefficient $(\chi^0 - \chi)$. The constant in (6.03) can be adjusted so as to make this coefficient vanish at a given point η_0 (between η and η', say). Comparing the expressions (3.16) and (6.04) for χ and χ^0 we see that the necessary condition is

$$\frac{a}{1+a\eta} = \frac{1}{M(\eta)} \frac{\mathrm{d}M(\eta)}{\mathrm{d}\eta} \qquad (\text{for} \quad \eta = \eta_0) \tag{6.10}$$

The solution takes the form

$$W = 2\pi i \sqrt{\left[\frac{(1+a\eta)}{(1+a\xi)M(\eta)}\right]} \sqrt{\left[\frac{(1+a\eta')}{(1+a\xi')M(\eta')}\right]} W_3 \tag{6.11}$$

where

$$W_3 = e^{-i\frac{2}{3}(\alpha^3-\alpha'^3)}\,\Psi(\zeta+\alpha-\alpha',\,\alpha^2,\,\alpha'^2,\,q) \tag{6.12}$$

We observe that if $\varkappa=0$ and $M(\eta)$ is a linear function of η then (6.10) holds everywhere (not only for $\eta=\eta_0$) and the transformation (6.03) is the same as (3.20). Thus in the case $\varkappa=0$ expressions (6.11) and (6.12) give the exact solution.

If the terms in $\partial/\partial\alpha$ and in $\chi^0-\chi$ are neglected in the operator D_3 and if the term corresponding to $(1/\beta)\,D_2$ in (3.24) is also neglected, we obtain for W_3 an equation

$$\frac{\partial^2 W_3}{\partial z^2}+2iR_\alpha W_3 = 0 \tag{6.13}$$

which is satisfied by the Fresnel integral.

The remarks at the end of Section 5 concerning infinite values and change of sign of the variable β are applicable to the variable α as well.

7. *Transformation Group for the Parabolic Equation*

The results obtained can be considered in conjunction with the transformation group for the parabolic equation. We have according to (3.07) and (3.19)

$$(D^2+ikR)W = 0 \tag{7.01}$$

where

$$D = \frac{\varrho(\eta)}{\xi-\eta}\frac{\partial}{\partial\eta}; \qquad R = 2\frac{\partial}{\partial\xi}+\frac{1}{\xi-\eta} \tag{7.02}$$

We make the transformation from ξ, η, W to new variables ξ^*, η^*, W^*, defined by

$$\xi^* = \frac{\xi}{1+a\xi}; \qquad \eta^* = \frac{\eta}{1+a\eta} \tag{7.03}$$

$$W(\xi,\eta) = \frac{1}{\sqrt{(1+a\xi)}}\,W^*(\xi^*,\eta^*) \tag{7.04}$$

and we use the notation

$$\varrho^*(\eta^*) = \frac{\varrho(\eta)}{(1+a\eta)^3} \tag{7.05}$$

Then equation (7.01) transforms into

$$(D^{*2}+ikR^*)W^* = 0 \tag{7.06}$$

where

$$D^* = \frac{\varrho^*(\eta^*)}{\xi^*-\eta^*}\,\frac{\partial}{\partial\eta^*}\,; \qquad R = 2\frac{\partial}{\partial\xi^*}+\frac{1}{\xi^*-\eta^*} \tag{7.07}$$

Thus the transformed equation is of the same form as the original one with the only difference that the function $\varrho(\eta)$ is replaced by $\varrho^*(\eta^*)$. Therefore, we have a family of cylinders depending on the parameter a such that the solution of the diffraction problem for one of them yields the solution for the other members of the family. From this standpoint it is clear why the case $\varkappa=0$ considered in Section 5 is reducible to the case of a circular cylinder ($\varrho=$constant) considered in Section 4.

The variable z is not affected by the transformation because

$$M^*(\eta^*) = \frac{M(\eta)}{1+a\eta} \tag{7.08}$$

and

$$\mathrm{d}z = \frac{k}{2}\,\frac{\mathrm{d}\eta}{M^2(\eta)} = \frac{k}{2}\,\frac{\mathrm{d}\eta^*}{M^{*2}(\eta^*)} \tag{7.09}$$

The variable ϑ^*, however, is not identical with ϑ since we have

$$\eta^*\,\mathrm{d}\vartheta^* = \eta\,\mathrm{d}\vartheta \tag{7.10}$$

We note that the quantity μ^2 defined by

$$\frac{1}{\mu^2} = \frac{2}{k}\,M(\eta)M(\eta')\,\frac{\xi-\xi'}{(\xi-\eta)\,(\eta'-\xi')} \tag{7.11}$$

(see (5.10)) is invariant with respect to the transformation (7.03).

The transformation (7.03) with a suitable choice of the parameter a can also be used as a preliminary stage of the solution of the diffraction problem (before the introduction of the variables γ or β). The variable α used in Section 6 in order to improve the solution in the Fresnel zone is simply $\alpha=\gamma^*$.

8. *Conclusion*

In the course of investigating the diffraction of short waves by a convex cylinder with slowly varying curvature we were led to three expressions (6.01), (5.07), and (6.11)–(6.12) which will be referred to as the γ-, β-, and α-expressions. The γ-expression is a rather crude one, especially near the cylinder surface and gives a relative error of the order χ (or $k^{-1/3}$ when $k\to\infty$) where it is applicable. The β-expression is valid in the first zone

(near the surface), in the *GD* region, and in a part of the second (Fresnel) zone (see Fig. 3). The α-expression is valid in the *GD* region and in the entire second zone. The approximation given by the β-expression is of the order $\varkappa$ or $k^{-2/3}$; for $\varkappa=0$ the α- and the β-expressions coincide and give a rigorous solution of the parabolic equation. For a circular cylinder all three expressions coincide.

The results obtained show that the parabolic equation in ray coordinates allows us to find an asymptotic solution of diffraction problems in cases when a rigorous solution cannot be expressed in finite terms. Further possibilities of improving the solution are offered by the application of the transformation given in Section 7.

PART II

Tropospheric Radiowave Propagation

CHAPTER 10

DIFFRACTION OF RADIO WAVES AROUND THE EARTH'S SURFACE†

Abstract. — The problem of the propagation of radio waves around the spherical surface of the earth is investigated. The diffraction effects are considered but the influence of the atmosphere is neglected. The aim of the paper is to derive formulae for the wave amplitude as a function of the elevation of the source, its distance from the point of observation (situated on the surface of the earth), of the wave length and of the electrical properties of the soil. The main result is the derivation of an expression for the attenuation factor in form of an integral. This expression is valid for all values of parameters, which are of practical interest. In the limiting cases the well known formulae are obtained: the Weyl–van der Pol formula for the illuminated region and the formula which corresponds to the first term in Watson's series for the shaded region (the latter in a slightly corrected form). Essentially new is the investigation of the region of the penumbra (near the line of horizon). Formulae are obtained which give a continuous transition from the illuminated region to the shaded one. Methods for the numerical calculation of sums and integrals involved in the problem are elaborated.

Introduction

There are many papers devoted to the problem of the diffraction of radio waves around the surface of the earth. A review of more recent investigations may be found in a paper by B. A. Vvedenski [20].

The interest in this problem is justified by the fact, that at small distances of the order of a few hundreds of kilometres, the refraction of radio waves in the atmosphere may be neglected and the most significant role in the propagation of radio waves is played by diffraction.

In spite of the fact that a rigorous solution of the problem of diffraction by a sphere had already been obtained some decades earlier, no practically suitable approximate solution has been proposed up to now. In this paper we intend to fill this gap.

† Fock, 1945.

1. *Statement of the Problem and its Solution in the Form of Series*

We denote by r, ϑ, φ spherical coordinates with origin at the centre of the earth.

The equation of the earth's surface (considered as even) is $r=a$, where a is the radius of the earth. Let us suppose that a vertical electric dipole is located at the point $r=b$, $\vartheta=0$ (where $b>a$). Suppressing the time-dependent factor $e^{-i\omega t}$ in the field components, we can express these components by means of the Hertz function U which depends on r and ϑ only. Denoting by k the absolute value of the wave vector we obtain for the field in the air:

$$\left.\begin{aligned} E_r &= \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right)\\ E_\vartheta &= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial\vartheta}\right)\\ H_\varphi &= -ik\frac{\partial U}{\partial\vartheta}\end{aligned}\right\} \tag{1.01}$$

the other components being equal to zero. Similar equations hold for the field in the earth.

For $r>a$ the function U satisfies the equation

$$\Delta U+k^2U = 0 \tag{1.02}$$

and the radiation condition at infinity

$$\lim_{z\to\infty}\left(\frac{\partial(rU)}{\partial r}-ikrU\right) = 0 \tag{1.03}$$

If $b>a$, i.e. if the source (dipole) is located *over* the earth's surface and not *on* the surface itself, U must have a singularity at the point $r=b$, $\vartheta=0$ such that

$$U = \frac{e^{ikR}}{R}+U^* \tag{1.04}$$

and U^* remains finite if $kR\to 0$. In this formula

$$R = \sqrt{(r^2+b^2-2rb\cos\vartheta)} \tag{1.05}$$

is the distance from the dipole. On the earth's surface the Hertz function U has to satisfy the boundary conditions which ensure the continuity of the tangential components E_ϑ and H_φ.

If we denote the Hertz function within the earth by U_2 these boundary conditions will have the form:

$$k^2 U = k_2^2 U_2; \qquad \frac{\partial}{\partial r}(rU) = \frac{\partial}{\partial r}(rU_2) \qquad \text{for} \qquad r = a \tag{1.06}$$

For $0 \leqslant r \leqslant a$ (within the earth) the function U_2 has to satisfy an equation similar to (1.02) and to remain finite.

The quantity k_2 in formula (1.06) and in subsequent formulae is determined by the equation

$$k_2^2 = \varepsilon k^2 + i\frac{4\pi\sigma}{c}k \tag{1.07}$$

and by the condition $\operatorname{Im}(k_2) > 0$. It is useful to introduce instead of the conductivity of the earth σ, a length l which characterizes the specific resistance of the earth. We put

$$l = c/(4\pi\sigma) \tag{1.08}$$

For sea water the values of l vary from 0·05 cm (very salty water) to 0·5 cm (water with little salt). For the soil this length is hundreds or thousands times greater. Introducing the complex dielectric constant of the earth

$$\eta = \varepsilon + i\frac{\lambda}{2\pi l} \tag{1.09}$$

we have

$$k_2 = k\sqrt{\eta} \tag{1.10}$$

The solution of our problem in the form of series is well known. We write down the necessary formulae, without giving their derivation.

We put

$$\begin{aligned} \psi_n(x) &= \sqrt{\left(\frac{\pi x}{2}\right)} \cdot J_{n+\frac{1}{2}}(x) \\ \zeta_n(x) &= \sqrt{\left(\frac{\pi x}{2}\right)} \cdot H^{(1)}_{n+\frac{1}{2}}(x) \end{aligned} \tag{1.11}$$

where $J_\nu(x)$ is the Bessel function and $H_\nu^{(1)}(x)$ is the Hankel function of the first kind. These functions are connected by the relation

$$\psi_n(x)\zeta_n'(x) - \psi_n'(x)\zeta_n(x) = i \tag{1.12}$$

We introduce a special notation for the logarithmic derivative of the

function $\psi_n(x)$:

$$\chi_n(x) = \frac{\psi_n'(x)}{\psi_n(x)}. \tag{1.13}$$

As seen from (1.01), the field on the earth's surface may be expressed by the quantities

$$U_a = U|_{r=a}; \qquad U_a' = \left[\frac{\partial}{\partial r}(rU)\right]_{r=a} \tag{1.14}$$

For these quantities the following series in Legendre polynomials may be obtained:

$$U_a = -\frac{1}{kab}\sum_{n=0}^{\infty}\frac{(2n+1)\zeta_n(kb)}{\zeta_n'(ka)-\frac{k}{k_2}\chi_n(k_2a)\zeta_n(ka)}P_n(\cos\vartheta) \tag{1.15}$$

$$U_a' = -\frac{k}{k_2b}\sum_{n=0}^{\infty}\frac{(2n+1)\zeta_n(kb)\chi_n(k_2a)}{\zeta_n'(ka)-\frac{k}{k_2}\chi_n(k_2a)\zeta_n(ka)}P_n(\cos\vartheta) \tag{1.16}$$

Our task is to perform an approximate summation of these series.

2. *The Summation Formula*

The sums we have to calculate are of the form

$$S = \sum_{\nu=1/2,\,3/2\ldots}\nu\varphi(\nu)P_{\nu-1/2}(\cos\vartheta) \tag{2.01}$$

In the sum (1.15) the function $\varphi(\nu)$ (disregarding a constant factor) is equal to

$$\varphi(\nu) = \frac{\zeta_{\nu-1/2}(kb)}{\zeta_{\nu-1/2}'(ka)-\frac{k}{k_2}\chi_{\nu-1/2}(k_2a)\zeta_{\nu-1/2}(ka)} \tag{2.02}$$

In the sum (1.16) this function differs from (2.02) by the factor $\chi_{\nu-1/2}(k_2a)$.

For the direct computation of the sum it would be necessary to take the number of the terms approximately equal to $2ka$, i.e. to twice the number of waves which may be put around the earth circumference. Since this number is enormous, it is evident that such a direct summation is impossible. For the calculation of the sum S it is necessary to make use of the fact that $\varphi(\nu)$ is an analytical function and to transform this sum into an integral, which is to be evaluated by some approximate method. Such a transformation was first proposed by Watson [18] in 1918 and was then

used by various authors. But all these authors aimed to bring the expression obtained by this transformation to the form of a sum of residues, while our aim is to separate out a main term which is easier to investigate and to estimate the magnitude of the remainder. The method of computation of the main term is not predetermined by this technique.

When performing our transformation we have to bear in mind the following general properties of the function $\varphi(\nu)$. It is an analytical function of ν meromorphic in the right half-plane. It has poles only in the first quadrant and is holomorphic in the fourth quadrant. It decreases at infinity in such a way that all the integrals considered converge.

The Legendre functions that enter (2.01) can be expressed by means of the function

$$G_\nu = \frac{\Gamma\left(\nu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(\nu+1)} F\left(\frac{1}{2};\ \frac{1}{2},\ \nu+1,\ \frac{i\,e^{-i\vartheta}}{2\sin\vartheta}\right) \tag{2.03}$$

where F denotes the hypergeometrical function. Denoting by G_ν^* and by $P_{\nu-1/2}^*$ the expressions which are obtained from G_ν and from $P_{\nu-1/2} = P_{\nu-\frac{1}{2}}(\cos\vartheta)$ by replacing ϑ by $\pi-\vartheta$ we get

$$P_{\nu-1/2} = \frac{1}{\pi\sqrt{(2\sin\vartheta)}}\left[e^{i\nu\vartheta-i\frac{\pi}{4}}G_\nu^* + e^{-i\nu\vartheta+i\frac{\pi}{4}}G_\nu\right] \tag{2.04}$$

It is seen from (2.03) that if the values of ν lie outside of a certain sector, which includes the negative real axis, and if $|\nu\sin\vartheta|$ is large, then the function G_ν (and also G_ν^*) is approximately equal to

$$G_\nu \sim \sqrt{(\pi/\nu)} \tag{2.05}$$

Substituting (2.05) into (2.04) we get the well-known asymptotic expression for $P_{\nu-1/2}$. If we denote the first term in formula (2.04) by $B(\nu)$:

$$B(\nu) = \frac{1}{\pi\sqrt{(2\sin\vartheta)}}\, e^{i\nu\vartheta-i\frac{\pi}{4}} G_\nu^* \tag{2.06}$$

the following relation may be proved:

$$P_{\nu-1/2}^* = e^{i(\nu-1/2)\pi} P_{\nu-1/2} + 2i\cos\nu\pi B(\nu) \tag{2.07}$$

We shall use this relation later on. We note that $B(\nu)$ is holomorphic in the right half-plane.

Let us consider in the plane of the complex variable ν three contours: (1) the loop C_1 which starts at infinity on the positive real axis, runs above

the real axis, encircles the origin counter-clockwise and returns to the starting point at infinity running below the real axis; (2) the broken line C_2, which contains the first quadrant and is described (in its horizontal part drawn slightly over the real axis) from the left to the right side; (3) the straight line C_3 which crosses the origin and is inclined at a small angle to the imaginary axis. This line is described from the top to the bottom and lies in the second and fourth quadrants.

We can write the sum S in the form

$$S = \frac{i}{2}\int_{C_1} \nu\varphi(\nu) \sec \nu\pi P^*_{\nu-1/2}\, \mathrm{d}\nu \tag{2.08}$$

since the integral on the right-hand side reduces to the sum of the residues at the points $\nu = n + {}^1/_2$. The function $\varphi(\nu)$ being holomorphic in the fourth quadrant, we may replace the contour C_1 by the contours C_2 and C_3 and write

$$S = -\frac{i}{2}\int_{C_2} \nu\varphi(\nu) \sec \nu\pi P^*_{\nu-1/2}\, \mathrm{d}\nu + \frac{i}{2}\int_{C_3} \nu\varphi(\nu) \sec \nu\pi P^*_{\nu-1/2}\, \mathrm{d}\nu \tag{2.09}$$

This transformation of the sum corresponds to the usual one; the integral along the contour C_3 is neglected because of the smallness of the odd part of $\varphi(\nu)$ (an estimate of its magnitude will be given below), and the integral along C_2 reduced to the sum of residues. But we shall go a step further and divide the integral along C_2 into two parts: the main term and the correction term. Inserting in the integral the expression (2.07) for $P^*_{\nu-1/2}$ we have

$$S = S_1 + S_2 + S_3 \tag{2.10}$$

where

$$S_1 = \int_C \nu\varphi(\nu) B(\nu)\, \mathrm{d}\nu \tag{2.11}$$

$$S_2 = -\frac{1}{2}\int_{C_2} \nu\varphi(\nu) \sec \nu\pi\, \mathrm{e}^{i\nu\pi} P_{\nu-1/2}\, \mathrm{d}\nu \tag{2.12}$$

$$S_3 = \frac{i}{2}\int_{C_3} \nu\varphi(\nu) \sec \nu\pi P^*_{\nu-1/2}\, \mathrm{d}\nu \tag{2.13}$$

The integrand in S_1 has no poles on the real axis (and also in the fourth quadrant). Therefore, there is no difference, whether we evaluate the integral S_1 along C_2 or along C_3. We have denoted by C any contour, which is equivalent to C_2 or C_3.

The representation of S as a sum of three integrals (2.10) is exact—nothing was neglected in our derivation. But the estimation of the magnitude of S_2 and S_3 shows that these integrals are negligibly small as compared to S_1. In fact, if we evaluate the integral S_2 as a sum of residues at the poles of $\varphi(\nu)$ we shall see that its ratio compared to S_1 is of the order

$$|\, e^{2i\nu_1(\pi-\vartheta)}\, | \tag{2.14}$$

where ν_1 is the pole of $\varphi(\nu)$ nearest to the real axis. The imaginary part of ν_1 is positive and for large values of ka will be equal to

$$Im\,(\nu_1) = c(ka)^{1/3} \tag{2.15}$$

where c is a pure number of the order of unity (for a perfect conductor $c=0{\cdot}70$). Since ka is very large, of the order of a million (for $\lambda=40$ m, $ka=10^6$), it is clear, that the quantity (2.15) will be large (for instance, equal to 70) and the quantity (2.14) will be negligibly small. (In our problem ϑ cannot reach the value π since in this case we have to take into account the influence of ionized layers of the atmosphere and our formulae cease to be valid.)

The value of the integral S_3 is determined by the odd part of $\varphi(\nu)$. But the odd part of this function will be of the order

$$|\, e^{2ik_2a}\, | \tag{2.16}$$

Since the imaginary part of k_2a is positive and very large, the value of (2.16) will be extremely small.

The following physical picture gives a notion of the smallness of the integrals S_2 and S_3. The integral S_3 is the amplitude of a wave which travelled once or several times around the globe without suffering refraction (by means of diffraction only). The integral S_3 is the amplitude of a wave which traversed a sphere with the diameter of the globe and with an absorption coefficient equal to that of the earth. It is clear that both integrals are negligibly small as compared with the amplitude of the wave which reached the observer through the air by the shortest way.

Therefore with the highest physically possible accuracy (i.e. with an error which is negligibly small as compared with the errors involved because of the simplified physical model) the sum S defined by (2.01) may be put equal to the integral S_1 alone. This integral may be written in the form

$$S_1 = \frac{e^{i(\pi/4}}{\pi\sqrt{(2\sin\vartheta)}} \int_C \nu\varphi(\nu)\, e^{i\nu\vartheta}\, G_\nu^*\, d\nu \tag{2.17}$$

which follows from (2.11) when the expression (2.06) for $B(\nu)$ is inserted.

3. The Evaluation of the Hertz Function for the Illuminated Region

If $\varphi(\nu)$ is the function (2.02), then the relation between the sum S and the quantity U_a is

$$U_a = -\frac{2}{kab}S \tag{3.01}$$

Therefore, our approximate expression for U_a may be written

$$U_a = \frac{2\,e^{i\frac{3\pi}{4}}}{\pi kab\sqrt{(2\sin\vartheta)}}\int_C \nu\varphi(\nu)\,e^{i\nu\vartheta}\,G_\nu^*\,d\nu \tag{3.02}$$

The position of the main part of the integration path in (3.02) depends on the point at which the integral is evaluated. In general the main part is in the vicinity of the point $\nu=\nu_0$ where

$$\nu_0 = kh_c = k\frac{ab\sin\vartheta}{\sqrt{\{a^2+b^2-2ab\cos\vartheta\}}} \tag{3.03}$$

The quantity h_c is the length of the perpendicular dropped from the earth's centre onto the ray (i.e. on the straight line which connects the source and the point of observation).

For the approximate evaluation of the integral U_a it is necessary to obtain the asymptotic expressions for the functions G_ν^* and $\varphi(\nu)$ valid on the main part of the integration path. Since ν_0 and $\nu_0\vartheta$ are large as compared with unity, we may put according to (2.05)

$$G_\nu^* = \sqrt{(\pi/\nu)} \tag{3.04}$$

For the Hankel functions involved in $\varphi(\nu)$ one may tentatively use the Debye expression

$$\zeta_{\nu-1/2}(\varrho) = \frac{1}{\sqrt[4]{[1-(\nu^2/\varrho^2)]}}\cdot e^{i\left(\xi-\frac{\pi}{4}\right)} \tag{3.05}$$

where

$$\xi = \int_\nu^\varrho \sqrt{(1-\nu^2/\varrho^2)}\,d\varrho \tag{3.06}$$

These expressions are valid provided the condition

$$|\varrho^2-\nu^2| \gg \varrho^{4/3} \tag{3.07}$$

is satisfied. As to the function $\chi_{\nu-1/2}(k_2a)$, its value near the point $\nu=\nu_0$ may be represented with sufficient accuracy by the expression

$$\chi_{\nu-1/2}(k_2a) = -i\sqrt{\left(1-\frac{\nu^2}{k_2^2a^2}\right)} \tag{3.08}$$

In order to make clear in which cases the inequality (3.07) is satisfied, let us introduce the parameter

$$p = \left(\frac{ka}{2}\right)^{1/3}\cos\gamma \tag{3.09}$$

where γ is the angle between the vertical direction at the observation point and the direction from this point to the source.

It is easily seen that for $\nu=\nu_0$, $\varrho=ka$ the inequality (3.07) is equivalent to the condition that p should be large and positive. Such values of p correspond to the illuminated region. The values of p of the order of unity (positive and negative ones) correspond to the region of penumbra: the special value $p=0$ gives the boundary of the geometrical shadow (horizon line). Large and negative values of p correspond to the shadow region.

In this section we shall investigate the case of a large positive p (illuminated region); other cases will be investigated in the next sections.

We have seen that if $p \gg 1$ the Debye expressions for the Hankel functions are valid. Inserting these expressions into (3.02) and using (3.04) and (3.08) we get

$$U_a = \frac{2\,e^{i\frac{\pi}{4}}}{kab\sqrt{(2\pi\sin\vartheta)}}\sqrt{\left(\frac{b}{a}\right)}\cdot\int_C \sqrt[4]{\left(\frac{k^2a^2-\nu^2}{k^2b^2-\nu^2}\right)}\times$$

$$\times\frac{e^{i\omega}\sqrt{\nu}\cdot d\nu}{\sqrt{\left(1-\frac{\nu^2}{k^2a^2}\right)}+\frac{k}{k_2}\sqrt{\left(1-\frac{\nu^2}{k_2^2a^2}\right)}} \tag{3.10}$$

where

$$\omega = \int_{ka}^{kb}\sqrt{\left(1-\frac{\nu^2}{\varrho^2}\right)}\,d\varrho+\nu\vartheta \tag{3.11}$$

If the condition

$$kh\cos\gamma \gg 1 \tag{3.12}$$

is satisfied, where $h = b-a$ is the height of the source above the earth, the integral (3.10) can be calculated by means of the method of stationary

phase and the following "reflection formula" is obtained:

$$U_a = \frac{e^{ikR}}{R} W \ . \tag{3.13}$$

In this formula

$$R = \sqrt{(a^2+b^2-2ab\cos\vartheta)} \tag{3.14}$$

is the distance from the source and W is the "attenuation function" which in our case is equal to

$$W = \frac{2}{1+\frac{k}{k_2}\sqrt{\left(1-\frac{k^2}{k_2^2}\sin^2\gamma\right)}\cdot\sec\gamma} \tag{3.15}$$

The quantity U'_a defined by the series (1.16) differs (in our approximation) from U_a by a constant factor only. We have

$$U'_a = -\frac{ik^2a}{k_2}\sqrt{\left(1-\frac{k^2}{k_2^2}\sin^2\gamma\right)}\cdot U_a \tag{3.16}$$

The last formula is valid not only for the illuminated region, but also in other cases.

If condition (3.12) is not satisfied, the denominator in the integrand (3.10) cannot be considered as slowly varying. If instead of (3.12) we suppose that the conditions

$$1 \ll \frac{R^2}{h^2} \ll (ka)^{2/3} \tag{3.17}$$

$$1 \ll kR \ll \frac{a}{h} \tag{3.18}$$

are satisfied (the inequality $p \gg 1$, being a consequence of these conditions), the integral (3.10) can be approximately calculated by introducing a new integration variable μ, defined by

$$\mu = \sqrt{\left(1-\frac{\nu^2}{k^2a^2}\right)} \tag{3.19}$$

For the function W in (3.13) the following approximate expression is obtained:

$$W = e^{-i\frac{3\pi}{4}}\sqrt{\left(\frac{2kR}{\pi}\right)}\int_\Gamma e^{-i\frac{kR}{2}(\mu-\mu_0)^2}\frac{\mu\,d\mu}{\mu+\left(\frac{k}{k_2}\right)} \tag{3.20}$$

where

$$\mu_0 = \frac{h}{R} \tag{3.21}$$

is the inclination of the ray to the horizon. The contour Γ is a straight line which passes through the point $\mu = \mu_0$ moving there from the fourth to the second quadrant of the plane of μ (or of $\mu - \mu_0$ to be more exact). The integral (3.20) can be calculated without any further approximation and gives the well-known Weyl–van der Pol formula. If we put

$$\sigma = e^{i\frac{\pi}{4}} \frac{k}{k_2} \sqrt{\left(\frac{kR}{2}\right)}, \qquad \tau = e^{i\frac{\pi}{4}} \frac{h}{R} \sqrt{\left(\frac{kR}{2}\right)} \tag{3.22}$$

we shall have

$$W = 2 - 4\sigma\, e^{-(\sigma+\tau)^2} \int_{i\infty}^{\sigma+\tau} e^{\alpha^2}\, d\alpha \tag{3.23}$$

To obtain the field components from our expressions for U_a and U_a' we have to differentiate these expressions by ϑ which is easily done, since we may regard all factors in (3.13), except e^{ikR}, as constants.

4. *Asymptotic Expressions for the Hankel Functions*†

In the following we have to consider the case when the point of observation is in the region of the penumbra.

This case is characterized by the values of the parameter p (positive or negative) of the order unity. As the inequality (3.01) is not satisfied in this case, the Debye expressions (3.05) for the Hankel functions are not valid on the main part of the integration contour and must be replaced by some others. The new expressions for the Hankel functions suitable for our purpose can be obtained from the asymptotic expressions which are given in Ref. 19, or from the formulae given in the well-known treatise of Watson [18]; it is more simple, however, to deduce them independently.

Our aim is to find an approximate expression for the Hankel function in terms of the function $w(t)$, defined by the integral

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{tz - {}^1\!/\!_3 z^3}\, dz \tag{4.01}$$

the contour Γ running from infinity to the origin along the line $\arg z = -2\pi/3$ and from the origin to infinity along the line $\arg z = 0$ (the

† See also Appendix.

positive real axis). The function $w(t)$ satisfies the differential equation

$$w''(t) = tw(t) \tag{4.02}$$

with the initial conditions

$$\begin{aligned} w(0) &= \frac{2\sqrt{\pi}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} e^{\frac{i}{6}\pi} = 1{\cdot}0899290710 + i\,0{\cdot}6292708425 \\ w'(0) &= \frac{2\sqrt{\pi}}{3^{4/3}\Gamma(4/3)} e^{-\frac{i}{6}\pi} = 0{\cdot}7945704238 - i\,0{\cdot}4587454481 \end{aligned} \tag{4.03}$$

$w(t)$ is an integral transcendental function, which can be expanded into a power series of the form

$$\begin{aligned} w(t) = w(0)\left[1 + \frac{t^3}{2.3} + \frac{t^6}{(2.5)(3.6)} + \frac{t^9}{(2.5.8)(3.6.9)} + \dots\right] + \\ + w'(0)\left[t + \frac{t^4}{3.4} + \frac{t^7}{(3.6)(4.7)} + \frac{t^{10}}{(3.6.9)(4.7.10)} + \dots\right] \end{aligned} \tag{4.04}$$

If we separate in $w(t)$ the real and the imaginary parts (for real values of t) putting

$$w(t) = u(t) + iv(t) \tag{4.05}$$

then $u(t)$ and $v(t)$ will be two independent integrals of equation (4.02) connected by the relation

$$u'(t)v(t) - u(t)v'(t) = 1 \tag{4.06}$$

The asymptotic expressions for these functions for large negative values of t are obtained by separation of the real and imaginary parts in the formulae

$$w(t) = e^{i\frac{\pi}{4}}(-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} \tag{4.07}$$

$$w'(t) = e^{-i\frac{\pi}{4}}(-t)^{\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} \tag{4.08}$$

For large positive values of t the asymptotic expressions for $u(t)$, $v(t)$ and their derivatives are of the form

$$u(t) = t^{-\frac{1}{4}} e^{\frac{2}{3}t^{3/2}}; \qquad u'(t) = t^{1/4} e^{\frac{2}{3}t^{3/2}} \tag{4.09}$$

$$v(t) = \frac{1}{2} t^{-1/4} e^{-\frac{2}{3}t^{3/2}}; \qquad v'(t) = -\frac{1}{2} t^{1/4} e^{-\frac{2}{3}t^{3/2}} \tag{4.10}$$

From the series (4.04) the following relations are easily deduced:

$$w\left(t\,e^{i\frac{\pi}{3}}\right) = 2\,e^{i\frac{\pi}{6}}\,v(-t) \tag{4.11}$$

$$w\left(t\,e^{i\frac{2\pi}{3}}\right) = e^{i\frac{\pi}{3}}\,[u(t)-iv(t)] \tag{4.12}$$

These relations describe the behaviour of $w(t)$ in the complex t-plane.

We note that $w(t)$ is expressible in terms of the Hankel function of the order 1/3 according to the formula

$$w(t) = \sqrt{\left(\frac{\pi}{3}\right)}\cdot e^{i\frac{2\pi}{3}}\,(-t)^{1/2}H^{(1)}_{1/3}\left[\frac{2}{3}(-t)^{3/2}\right] \tag{4.13}$$

After having enumerated the main properties of $w(t)$, we now proceed to deduce the asymptotic expression for the Hankel function $H^{(1)}_\nu(\varrho)$ where ν and ϱ are large and nearly equal, so that the ratio

$$\frac{\nu-\varrho}{\sqrt[3]{(\varrho/2)}} = t \tag{4.14}$$

remains bounded, while ϱ tends to infinity.

The Hankel function $H^{(1)}_\nu(\varrho)$ admits the integral representation

$$H^{(1)}_\nu(\varrho) = \frac{1}{\pi i}\int_C e^{-\varrho\sinh v+\nu}\,dv \tag{4.15}$$

where the contour C consists of a part of the straight line $\mathrm{Im}(v) = -\pi$ described from $-\pi i-\infty$ to some point $v=v_0$ with $\mathrm{Re}(v_0)<0$ [e.g. $v_0 = (-\pi/\sqrt{3})-i\pi$], a straight line joining v_0 to the origin and, finally, the positive real axis described from the origin to infinity.

Let us express ν through t, according to (4.14), and introduce a new integration variable

$$z = \sqrt[3]{(\varrho/2)}\cdot v \tag{4.16}$$

Considering t and z as finite and ϱ as large, we can expand the integrand in (4.15) in a series of negative (fractional) powers of ϱ. Since the relevant part of the transformed contour C coincides with contour Γ we can write

$$H^{(1)}_\nu(\varrho) = \frac{1}{\pi i}\left(\frac{\varrho}{2}\right)^{-1/3}\int_\Gamma e^{tz-\frac{1}{3}z^3}\left[1-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}z^5+\dots\right]dz \tag{4.17}$$

and evaluate the integral using (4.01). We thus obtain

$$H^{(1)}_\nu(\varrho) = -\frac{i}{\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^{-1/3}\left[w(t)-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}w^{(5)}(t)+\dots\right] \tag{4.18}$$

By virtue of the differential equation (4.02) the fifth derivative equals

$$w^{(5)}(t) = t^2 w'(t) + 4tw(t) \tag{4.19}$$

Inserting this in (4.18) and using (1.11) we get the following expression for the function $\zeta_{\nu-1/2}(\varrho)$:

$$\zeta_{\nu-1/2}(\varrho) = -i\left(\frac{\varrho}{2}\right)^{1/6}\left\{w(t) - \frac{1}{60}\cdot\left(\frac{\varrho}{2}\right)^{-2/3}[t^2 w'(t) + 4tw(t)] + \ldots\right\} \tag{4.20}$$

Differentiating this expression with respect to ϱ (taking account of the dependence of t on ϱ with ν constant) we get the following expression for the derivative:

$$\zeta'_{\nu-\frac{1}{2}}(\varrho) = i\left(\frac{\varrho}{2}\right)^{-1/6}\left\{w'(t) - \frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}[(t^3+9)w(t) - 4tw'(t)] + \ldots\right\} \tag{4.21}$$

These expressions will be used in the next section.

5. *The Expressions of the Hertz Function Valid in the Penumbra Region*

We rewrite the expression (3.02) for the Hertz function replacing the quantity G_ν^* by its approximate value $\sqrt{(\pi/\nu)}$ and the quantity $\sin\vartheta$ before the integral by ϑ. We get

$$U_a = \frac{2\,e^{i\frac{3\pi}{4}}}{kab\sqrt{(2\pi\vartheta)}}\int_C \varphi(\nu)\,e^{i\nu\vartheta}\sqrt{\nu}\cdot d\nu \tag{5.01}$$

The contour C may be taken as identical with the contour C_2, which was defined in Section 2, or it may be replaced by some contour equivalent to C_2. The main part of the integration path lies in our case (i.e. for finite values of the parameter p) near the point $\nu = ka$. Consequently, the function $\chi_{\nu-1/2}(k_2 a)$ involved in (2.02) can be replaced by the value of (3.08) for $\nu = ka$. Introducing this in $\varphi(\nu)$ we obtain

$$\varphi(\nu) = \frac{\zeta_{\nu-1/2}(kb)}{\zeta'_{\nu-1/2}(ka) + i\dfrac{k}{k_2}\sqrt{\left(1-\dfrac{k^2}{k_2^2}\right)}\,\zeta_{\nu-1/2}(ka)} \tag{5.02}$$

For $\zeta_{\nu-1/2}$ and its derivative we must use expressions valid near the point $\nu = ka$. Such expressions were obtained in the preceding section.

Retaining in (4.20) and (4.21) the principal terms only, we get

$$\zeta_{\nu-1/2}(ka) = -i\left(\frac{ka}{2}\right)^{1/6} w(t) \tag{5.03}$$

$$\zeta'_{\nu-1/2}(ka) = i\left(\frac{ka}{2}\right)^{-1/6} w'(t) \tag{5.04}$$

where the variable t is connected with ν by the relation

$$\nu = ka+\left(\frac{ka}{2}\right)^{1/3} t \tag{5.05}$$

The numerator in (5.02) is obtained from (5.03) by replacing a by b and t by t', where t' is defined by

$$\nu = kb+\left(\frac{kb}{2}\right)^{1/3} t' \tag{5.06}$$

Equating (5.05) and (5.06) we obtain the connection between t and t'. Since the ratio h/a, where $h = b-a$, is small (we shall consider it of the same order as $(ka)^{-2/3}$) we must neglect it as compared to unity. We may then put

$$t' = t-y \tag{5.07}$$

where

$$y = \frac{kh}{(ka/2)^{1/3}} \tag{5.08}$$

is a quantity proportional to the height of the source over the earth's surface. We may call y the reduced height of the source. Hence, neglecting terms of the order h/a or $(ka)^{-2/3}$ we have

$$\zeta_{\nu-1/2}(kb) = -i\left(\frac{ka}{2}\right)^{1/6} w(t-y) \tag{5.09}$$

where t is determined by (5.05). (We have also replaced b by a in the factor before w).

Substitution of (5.03), (5.04) and (5.09) in (5.02) gives the desired approximate expression for $\varphi(\nu)$.

If we put for the sake of brevity

$$q = i\left(\frac{ka}{2}\right)^{1/3} \frac{k}{k_2} \sqrt{\left(1-\frac{k^2}{k_2^2}\right)} \tag{5.10}$$

we obtain

$$\varphi(\nu) = -\left(\frac{ka}{2}\right)^{1/3} \frac{w(t-y)}{w'(t)-qw(t)} \tag{5.11}$$

Remembering formulae (1.09) and (1.10), we may write for the quantity q

$$q = i\left(\frac{\pi a}{\lambda}\right)^{\frac{1}{3}} \frac{\sqrt{\left(\varepsilon-1+i\dfrac{\lambda}{2\pi l}\right)}}{\varepsilon+i\cdot\dfrac{\lambda}{2\pi l}} \tag{5.12}$$

or with the same accuracy

$$q = i\left(\frac{\pi a}{\lambda}\right)^{1/3} \cdot \frac{1}{\sqrt{\left(\varepsilon+1+i\dfrac{\lambda}{2\pi l}\right)}} \tag{5.13}$$

This form is slightly more convenient for calculations.

We have now to insert the value of $\varphi(\nu)$ from (5.11) into (5.01) and introduce the integration variable t. Making this substitution, we may replace the quantity $\sqrt{\nu}$ in the integrand by the constant value $\sqrt{(ka)}$ and also write b instead of a in the factor before the integral. The resulting formula can be written in the form

$$U_a = \frac{e^{ika\vartheta}}{a\vartheta} e^{-i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_C e^{ixt} \frac{w(t-y)}{w'(t)-qw(t)} \mathrm{d}t \tag{5.14}$$

where x denotes the quantity

$$x = \left(\frac{ka}{2}\right)^{1/3} \vartheta \tag{5.15}$$

which may be called the reduced horizontal distance from the source, while y and q have the values given by (5.08) and (5.13). The contour C must be such that all the poles of the integrand are enclosed within the contour; as we shall see later, they are all situated in the first quadrant of the t-plane. Thus we can carry out the integration in (5.14) from $i\infty$ to 0 and from 0 to $+\infty$.

In order to get a more clear idea on the ratio of the horizontal and the vertical scale of the variables x and y, let us write the expression for the parameter p, as defined by (3.09), in terms of x and y. From the consideration of the triangle with vertices at the earth's centre, at the source point and at the point of observation, the following approximate expression is

easily deduced:

$$p = \left(\frac{ka}{2}\right)^{1/3} \cos\gamma = \frac{y-x^2}{2x} \tag{5.16}$$

It follows that the equation of the horizon line is $x=\sqrt{y}$. Further we shall need the relation between the distance R from the source as measured along a straight line and the horizontal distance $a\vartheta$ as measured along the arc of a great circle. Assuming $a\vartheta \gg h$, i.e. $(ka)^{1/3}x \gg y$ this relation may be written

$$kR = ka\vartheta + \omega_0 \tag{5.17}$$

where

$$\omega_0 = \frac{y^2}{4x} + \frac{xy}{2} - \frac{x^3}{12} \tag{5.18}$$

6. *Discussion of the Expression for the Hertz Function*

The expression obtained for the Hertz function is most conveniently written in the form

$$U_a = \frac{e^{ika\vartheta}}{a\vartheta} V(x, y, q) \tag{6.01}$$

where

$$V(x, y, q) = e^{-i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int \frac{e^{ixt} w(t-y)}{w'(t) - q w(t)} \, dt \tag{6.02}$$

The quantity V may be called the attenuation factor by analogy with the quantity W, which was introduced earlier (see (3.13)). Let us determine the connection between V and W. Since in the denominators of expressions (3.13) and (6.01) the quantities R and $a\vartheta$ can be considered as equal, it follows from (5.17)

$$W = V e^{-i\omega_0} \tag{6.03}$$

We have now to investigate the expression (6.02) for V. We shall first consider the case of large positive values of p (illuminated region). This case has been already discussed using another method (Section 3). But, as formula (6.02) was obtained for the case of a finite p, it seems to be of interest to verify that it is also valid in the case of large p. If $p \gg 1$, the integration path may be deformed so as to cross the point where $\sqrt{(-t)} = p$. Its main part will be situated in the domain of large negative values of t, where expressions (4.07) and (4.08) for w and w' are applicable. Using

them and applying the method of stationary phase, we obtain

$$V = e^{i\omega_0} \cdot \frac{2p}{p-iq}. \tag{6.04}$$

and by virtue of (6.03)

$$W = \frac{2p}{p-iq} \tag{6.05}$$

The latter expression practically coincides with (3.15). We note that in the case when x is of the order of unity or large, the condition $p \gg 1$ is sufficient for the applicability of the method of stationary phase. If x is small, the further condition $y^2 \gg 2x$ is necessary. If the latter condition is not satisfied, but the inequality

$$x \ll y \ll \frac{1}{x} \tag{6.06}$$

is satisfied instead, the integral can be calculated by another method. Further simplifications of the asymptotic expression for $w(t-y)$ can then be made, and the integral (6.02) reduces to the form

$$V = e^{i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \cdot \int_C \frac{e^{ixt+iy\sqrt{(-t)}}}{\sqrt{(-t)}-iq} dt \tag{6.07}$$

Taking $\sqrt{(-t)}$ as integration variable, we are led to an integral of the form (3.20) $\left(\text{with } \sqrt{(-t)} = \left(\frac{ka}{2}\right)^{1/3} \mu\right)$ and we get again the Weyl–van der Pol formula (3.23) with the following values of σ and τ:

$$\sigma = e^{-i\frac{\pi}{4}} q\sqrt{x}; \qquad \tau = e^{i\frac{\pi}{4}} \frac{y}{2\sqrt{x}} \tag{6.08}$$

These values practically coincide with (3.22).

Let us now investigate the most interesting case when p is of the order of unity (positive or negative). We know that this is the region of the penumbra, where the diffraction effects play the dominant part.

If the values of x and y are of the order of unity, the most effective method of evaluation of the integral (6.02) is the representation of this integral in the form of a sum of residues taken at the poles of the integrand.

Denoting by $t_s = t_s(q)$ the roots of the equation

$$w'(t) - qw(t) = 0 \tag{6.09}$$

we obtain

$$V(x, y, q) = e^{i\frac{\pi}{4}} 2\sqrt{(\pi x)} \sum_{s=1}^{\infty} \frac{e^{ixt_s}}{t_s - q^2} \frac{w(t_s - y)}{w(t_s)} \tag{6.10}$$

The roots $t_s(q)$ are functions of the complex parameter q. For the value $q=0$ they reduce to the roots $t_s' = t_s(0)$ of the derivative $w'(t)$ and for $q=\infty$ they reduce to the roots $t_s^0 = t_s(\infty)$ of the function $w(t)$. The phases of t_s' and t_s^0 are equal to $\pi/3$, so that

$$t_s' = |t_s'| e^{i\frac{\pi}{3}}; \qquad t_s^0 = |t_s^0| e^{i\frac{\pi}{3}}$$

We give here the moduli of the first five roots t_s' and t_s^0:

s	$\lvert t_s' \rvert$	$\lvert t_s^0 \rvert$
1	1·01879	2·33811
2	3·24820	4·08795
3	4·82010	5·52056
4	6·16331	6·78671
5	7·37218	7·99417

For large values of s we have approximately

$$|t_s'| \cong \left[\frac{3\pi}{2}\left(s - \frac{3}{4}\right)\right]^{2/3}; \qquad |t_s^0| \cong \left[\frac{3\pi}{2}\left(s - \frac{1}{4}\right)\right]^{2/3} \tag{6.12}$$

To calculate the roots for finite values of q we may use the differential equation

$$\frac{dt_s}{dq} = \frac{1}{t_s - q^2} \tag{6.13}$$

which can be easily derived from (4.02). The root $t_s(q)$ is determined either as that solution of (6.13) which at $q=0$ reduces to t_s' or as that solution which at $q=\infty$ reduces to t_s^0. Both definitions are equivalent. Starting from the first definition, a series in ascending powers of q may be easily constructed for t_s; this series will converge for $|q| < |\sqrt{t_s}|$. Starting from the second definition we may construct a series in descending (negative) powers of q; this will converge for $|q| > |\sqrt{t_s}|$. These series will not be written down here. It may be noticed that the value of t, which for large values of $|q|$ is close to q^2, is not a root of equation (6.09).

If the condition $y^2 \ll 2|\sqrt{t_s}|$ is satisfied, we have the approximate relation

$$\frac{w(t_s-y)}{w(t_s)} = \cosh(y\sqrt{t_s}) - \frac{q}{\sqrt{t_s}}\sinh(y\sqrt{t_s}) \tag{6.14}$$

This relation permits us to estimate the value of remote terms in the series (6.10). If s is so large that $|q| \ll |\sqrt{t_s}|$ we have approximately $t_s = t_s(0) = t_s'$. It follows from this and from expression (6.14) that the series (6.10) is always convergent. But if x is small or if y is large, the series converges slowly, and to calculate its sum a large number of terms may be required.

In the shadow region, where p is large and negative, the series (6.10) converges very rapidly and its sum approximately reduces to its first term.

Our series (6.10) corresponds to that of Watson but has the advantage of simplicity.

The fundamental formula (6.02) permits us to investigate not only the limiting cases: large positive values of p (illuminated region), large negative values of p (shadow region), but also the intermediate cases, namely the region of the penumbra. While in the limiting cases our formula leads to an improvement of formulae previously known (the reflection formula and the Weyl–van der Pol formula for the illuminated region and the Watson series for the shadow region), in the transitional penumbra region it yields essentially new results.

The case when x and y are large and p is finite (short waves, penumbra) is of special interest. This case has not been investigated before as the known formulae are not valid here. In what follows we shall derive approximate formulae, which allow a complete discussion of this case.

We introduce the quantity

$$z = x - \sqrt{y} \tag{6.15}$$

which represents the reduced distance measured from the boundary of the geometrical shadow (and not from the source). In the region of geometrical shadow we have $z > 0$, in the visible region $z < 0$. Our parameter p, expressed in terms of z and x, takes the form

$$p = \frac{y-x^2}{2x} = -z + \frac{z^2}{2x} \tag{6.16}$$

In our case x is large and z is finite; hence we have approximately $p = -z$.

The main part of the integration path in (6.02) corresponds now to values of t of the order of unity; but if y is large and t finite we may use

for $w(t-y)$ the asymptotic expression (4.07) which gives

$$w(t-y) = e^{i\frac{\pi}{4}}(y-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(y-t)^{3/2}} \tag{6.17}$$

or approximately

$$w(t-y) = e^{i\frac{\pi}{4}} y^{-1/4} e^{i\frac{2}{3}y^{2/3}-i\sqrt{y}\cdot t} \tag{6.18}$$

Inserting (6.18) into (6.02) and replacing in the factor before the integral the quantity $x^{1/2}y^{-1/4}$ by unity, we get

$$V(x, y, q) = e^{i\frac{2}{3}y^{2/3}} V_1(x-\sqrt{y}, q) \tag{6.19}$$

where

$$V_1(z, q) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{izt}}{w'(t)-qw(t)}\, dt \tag{6.20}$$

The terms neglected in (6.19) are (for a finite z) of the order of $1/\sqrt{y}$ (or of $1/x$).

Therefore, the function $V(x, y, q)$ of two arguments x ,y, and of the parameter q reduces in our case to a function $V_1(z, q)$ of a single argument z and of the same parameter q. The resulting simplification is quite considerable.

Let us now derive the relation connecting the attenuation function W with the function V_1. We have the identity

$$\frac{2}{3}y^{3/2} = \omega_0 + \frac{1}{3}z^3 - \frac{z^4}{4x} \tag{6.21}$$

where ω_0 has the value (5.18). Omitting the last term in (6.21) we obtain from (6.03) and (6.19)

$$W = e^{\frac{i}{3}z^3} V_1(z, q) \tag{6.22}$$

Thus, in our approximation, the function W depends on x and y only through $z=x-\sqrt{y}$.

The function $V_1(z, q)$ is an integral transcendental function of the variable z. For a positive z we can evaluate the integral (6.20) as a sum of residues, and we get

$$V_1(z, q) = i2\sqrt{\pi} \sum_{s=1}^{\infty} \frac{e^{izt_s}}{(t_s-q^2)w(t_s)} \tag{6.23}$$

$$(\text{for } z > 0)$$

where t_s are the roots of equation (6.09) which were discussed earlier, the larger z, the more rapidly the series (6.23) converges. For a sufficiently large positive z its sum reduces to the first term. For finite negative values of z (e.g. $-2 < z < 0$) the integral (6.20) has to be evaluated numerically.

For large negative values of z this integral may be evaluated by the method of the stationary phase, and we get

$$V_1(z, q) = \frac{2\,e^{-\frac{i}{3}z^3}}{1+\frac{iq}{z}} \tag{6.24}$$

According to (6.22), this gives

$$W = \frac{2}{1+\frac{iq}{z}} \tag{6.25}$$

Since approximately $z = -p$, this coincides with expression (6.05).

We note in conclusion that our fundamental formula (6.02) can be obtained by the method of parabolic equations, proposed by M. Leontovich and applied by him [21] to the derivation of the Weyl–van der Pol formula. The application of Leontovich's method (in a slightly improved form) to our problem will be given in a separate paper.

CHAPTER 11

SOLUTION OF THE PROBLEM OF PROPAGATION OF ELECTROMAGNETIC WAVES ALONG THE EARTH'S SURFACE BY THE METHOD OF PARABOLIC EQUATIONS†

Abstract — The problem of propagation of electromagnetic waves along the surface of the earth is solved by the method of parabolic equations proposed by Leontovich. In the first section the surface of the earth is considered as plane and the well-known Weyl–van der Pol formula is deduced. This formula turns out to be the exact solution of the parabolic equation with corresponding boundary conditions. In the second section the surface is considered as spherical, and the resulting formula coincides with that obtained by Fock by the method of summation of infinite series which represents the rigorous solution of the problem.

A new form of the solution of the problem of propagation of electromagnetic waves from a vertical elementary dipole situated at a given height above the spherical surface of the earth was given in a paper by Fock [10]. In this solution the field is calculated for points on the surface of the earth, but according to the reciprocity theorem the same solution gives directly the field at any point above the surface if the dipole is located on the surface itself. In the present paper it is shown that Fock's solution can also be obtained by another method, namely by reducing the problem to an equation of the parabolic type for the "attenuation function".

The method of parabolic equations was proposed by Leontovich and applied by him to the solution of the same problem for the case of a plane earth. Since the considerations of the original paper by Leontovich [21] need some modifications, we shall give in what follows a new exposition of the method, applying it firstly to the case of a plane earth and considering then the case of a spherical earth.

† Leontovich and Fock, 1946.

1. *The Case of a Plane Earth*

We assume the time-dependence of all the field components to be of the form $e^{-i\omega t}$. In the following this factor will be omitted.

Let us denote by k the absolute value of the wave vector and by η the complex dielectric constant of the earth:

$$k = \frac{2\pi}{\lambda}; \qquad \eta = \varepsilon + i\frac{4\pi\sigma}{\omega} = \varepsilon + \frac{i}{kl} \tag{1.01}$$

The quantity

$$l = \frac{c}{4\pi\sigma}$$

having the dimensions of a length characterizes the specific resistance of the earth (this length varies from some tenths of a centimetre for sea water to ten and more metres for dry soil). Let U be the vertical component of the Hertz vector (the Hertz function). This function satisfies the equation

$$\Delta U + k^2 U = 0 \tag{1.03}$$

We shall write the Hertz function in the form

$$U = \frac{e^{ikR}}{R} \cdot W \tag{1.04}$$

where R is the distance from the point of observation to the source and the factor W is the so-called "attenuation function". As known, for $kR \to 0$ the Hertz function tends to infinity in such a way that W takes a finite value. We normalize W in such a manner that this value shall be equal to unity (it being supposed that both the source and the observation point remain *above* the surface of the earth).

In the following we assume, however, that the source is located *on* the earth's surface. Let us introduce cylindrical coordinates r, z with the origin in the dipole and the z-axis drawn vertically upwards. On the earth's surface we have $z=0$. The distance R will be $R=\sqrt{(r^2+z^2)}$. The principal "large parameter" of our problem is the quantity $|\eta|$. For large $|\eta|$ the attenuation function W is a slowly varying function of coordinates. In order to characterize the slowness of its variation it is useful to introduce the dimensionless cordinates

$$\varrho = \frac{kr}{2|\eta|}; \qquad \zeta = \frac{kz}{\sqrt{|\eta|}} \tag{1.05}$$

and to consider W as a function of ϱ and ζ. The derivatives of W with respect to its arguments will be then of the same order of magnitude as the function W itself.

Substitution of (1.04) into equation (1.03) gives for the function $W(\varrho, \zeta)$ an equation, which can be simplified if one supposes that the inclination angle of the ray to the horizon is small and that the distance from the source is at least equal to several wave lengths. These assumptions yield the inequalities

$$\frac{z}{r} \ll 1; \qquad kR \gg 1 \tag{1.06}$$

which are equivalent to

$$\frac{\zeta}{\varrho} \ll 2\sqrt{|\eta|}; \qquad \varrho \gg \frac{1}{2|\eta|} \tag{1.07}$$

Since $|\eta|$ is assumed to be large, the inequalities (1.07) hold over a wide range of values of ϱ and ζ (and certainly for values of ϱ and ζ of the order of unity). If the inequalities (1.07) are valid, the equation for $W(\varrho, \zeta)$ assumes the form

$$\frac{\partial^2 W}{\partial \zeta^2} + i\left(\frac{\partial W}{\partial \varrho} + \frac{\zeta}{\varrho}\frac{\partial W}{\partial \zeta}\right) = 0 \tag{1.08}$$

The terms omitted in (1.08) are of the order of $1/|\eta|$ as compared with those retained.

The boundary condition for W on the earth's surface is obtained from the condition for the Hertz vector

$$\frac{\partial U}{\partial z} = -\frac{ik}{\sqrt{\eta}}U \qquad (\text{for } z = 0) \tag{1.09}$$

given by Leontovich. It has the form

$$\frac{\partial W}{\partial \zeta} + q_1 W = 0 \qquad (\text{for } \zeta = 0) \tag{1.10}$$

where

$$q_1 = i\sqrt{\left(\frac{|\eta|}{\eta}\right)} = e^{i\frac{\pi-\delta}{2}} \tag{1.11}$$

and δ is the so-called loss angle, defined by

$$\delta = \operatorname{arc\,tan}\frac{1}{kl\varepsilon}; \qquad 0 < \delta < \frac{\pi}{2} \tag{1.12}$$

In the limit $|\eta| \to \infty$ the range of the variations of ϱ and ζ is $0 < \varrho < \infty$; $0 < \zeta < \infty$.

As a "condition at infinity" we may require that the function W should remain slowly varying such that the Hertz vector U satisfies the radiation condition. We may also require that for all positive values of ϱ and ζ (with the possible exception of the singular point $\varrho = 0$ of equation (1.08)) the function W should be bounded.

We now proceed to the formulation of the condition for $\varrho = 0$; since this is a point of some intricacy, we shall discuss it in a more detailed way.

We must state, firstly, that in the region close to the source, i.e., for small values of kR, the inequalities (1.07) cease to be satisfied; the differential equation (1.08) and the expression for W deduced from it become invalid. The region of small kR is a "forbidden zone" for our approximate function W. Therefore, the character of the singularity of the exact Hertz function cannot be used for the purpose of obtaining the required condition at $\varrho = 0$. For the statement of this condition we have to consider the properties of the Hertz function for large values of kR.

It is known that for large values of kR the so-called "reflection formula" may be used. This formula gives an approximation for the Hertz function in the whole space above the earth's surface, where the inclination of the ray to the horizon is not very small. If the Hertz function is normalized as stated above, the reflection formula may be written

$$U = (1+f)\frac{e^{ikR}}{R} \tag{1.13}$$

where

$$f = \frac{\eta \cos\gamma - \sqrt{(\eta - \sin^2\gamma)}}{\eta \cos\gamma + \sqrt{(\eta - \sin^2\gamma)}} \tag{1.14}$$

is the Fresnel coefficient (γ is the angle of incidence and $\cos\gamma = z/R$ in our case). The reflection formula is certainly valid in the region where the inequalities

$$1 \ll \frac{kz^2}{2r} \ll kr \tag{1.15}$$

are satisfied.

If $|\eta|$ is large and if

$$\frac{1}{\sqrt{|\eta|}} \ll \frac{z}{r} \ll 1 \tag{1.16}$$

then the Fresnel coefficient f is close to unity, and we have

$$U = 2\frac{e^{ikR}}{R} \tag{1.17}$$

When expressed in dimensionless coordinates ϱ, ζ the inequalities (1.15) and (1.16), which are necessary for formula (1.17) to be valid, become

$$1 \ll \frac{\zeta^2}{4\varrho} \ll 2\,|\eta|\,\varrho \tag{1.18}$$

$$1 \ll \frac{\zeta}{\varrho} \ll 2\sqrt{|\eta|} \tag{1.19}$$

To obtain the required condition for W at $\varrho \to 0$ we must carry out a double limiting process: firstly $|\eta| \to \infty$ and then $\varrho \to 0$. In the limit $|\eta| \to \infty$ the right-hand sides of the inequalities may be dropped and we get

$$1 \ll \frac{\zeta^2}{4\varrho}; \qquad 1 \ll \frac{\zeta}{\varrho} \tag{1.20}$$

If these relations are satisfied, the Hertz function tends to (1.17) and then

$$W \to 2 \tag{1.21}$$

Inequalities (1.20) are valid particularly for $\varrho \to 0$ if $\zeta > 0$. Hence the desired solution of (1.08) has to satisfy the condition

$$|W-2| \to 0 \qquad \text{for} \qquad \varrho \to 0 \qquad \text{and} \qquad \zeta > 0 \tag{1.22}$$

However, since $\varrho = 0$ is a singular point of the equation for W, condition (1.20) turns out to be insufficient for the unique determination of the solution. We replace it, therefore, by a more stringent condition

$$\left|\frac{W-2}{\sqrt{\varrho}}\right| \to 0 \qquad \text{for} \qquad \varrho \to 0 \qquad \text{and} \qquad \zeta > 0 \tag{1.23}$$

which is, as it will be seen later, a sufficient one.

Thus, for the determination of the "attenuation function" W we have the differential equation (1.08), the boundary conditions (1.10) and (1.23) and the condition of finiteness in the region considered (for $\varrho > 0$).

To simplify the differential equation, we make the substitution

$$W = \sqrt{\varrho} \cdot e^{-i\frac{\zeta^2}{4\varrho}}\, W_1 \tag{1.24}$$

Then the equation takes the form

$$\frac{\partial^2 W_1}{\partial \zeta^2}+i\frac{\partial W_1}{\partial \varrho} = 0 \tag{1.25}$$

The boundary condition for W_1 will be

$$\frac{\partial W_1}{\partial \zeta}+q_1 W_1 = 0 \qquad (\text{for } \zeta = 0) \tag{1.26}$$

The condition at $\varrho=0$ becomes

$$\left| W_1-\frac{2}{\sqrt{\varrho}}e^{i\frac{\zeta^2}{4\varrho}}\right| \to 0 \qquad (\text{for } \varrho \to 0) \tag{1.27}$$

Since $\varrho=0$ is a regular point of the equation for W_1 (in contrast to the equation for W) condition (1.27) is a sufficient one.

Solving (1.25) by means of separation of the variables, we easily obtain a particular solution which satisfies the boundary condition (1.26), namely

$$W_1 = e^{-i\nu^2\varrho}\left(\cos \nu\zeta-\frac{q_1}{\nu}\sin \nu\zeta\right) \tag{1.28}$$

where ν is the parameter of separation.

For real values of ν this expression remains finite and satisfies all conditions with the exception of (1.27). For complex values of ν (except the case $\nu=\pm iq_1$) expression (1.28) becomes infinite when $\zeta\to\infty$ and therefore, does not satisfy the necessary conditions. If $\nu=\pm iq_1$ this expression transforms into the form

$$W_1 = e^{iq_1^2\varrho-q_1\zeta} \tag{1.29}$$

According to (1.11) and (1.12), we have

$$\frac{\pi}{4} < \operatorname{arc} q_1 < \frac{\pi}{2} \tag{1.30}$$

and, consequently,

$$\operatorname{Re}(q_1) > 0; \qquad \operatorname{Re}(iq_1^2) < 0 \tag{1.31}$$

It follows that the real parts of the coefficients ϱ and ζ in (1.29) are negative and expression (1.28) also satisfies all conditions with the exception of (1.27).

In order to satisfy also the last condition, we construct a function which

is a superposition of solutions of the two forms (1.28) and (1.29)

$$W_1 = \int_0^\infty e^{-i\nu^2\varrho}\left(\cos\nu\zeta - \frac{q_1}{\nu}\sin\nu\zeta\right)f(\nu)\,d\nu + A\,e^{iq_1^2\varrho - q_1\zeta} \tag{1.32}$$

As easily seen, the singularity of W_1 for $\varrho \to 0$ is determined by the behaviour of $f(\nu)$ for large values of ν. The required singularity can be represented by the integral

$$\frac{4}{\sqrt{\pi}}e^{i\frac{\pi}{4}}\int_0^\infty e^{-i\nu^2\varrho}\cos\nu\zeta\,d\nu = \frac{2}{\sqrt{\varrho}}e^{i\frac{\zeta^2}{4\varrho}} \tag{1.33}$$

It is clear, therefore, that at infinity the function $f(\nu)$ tends to a finite limit equal to the constant factor in front of the integral in (1.33). Let us separate out in (1.32) the term

$$W_1^0 = \frac{4}{\sqrt{\pi}}e^{i\frac{\pi}{4}}\int_0^\infty e^{-i\nu^2\varrho}\left(\cos\nu\zeta - \frac{q_1}{\nu}\sin\nu\zeta\right)d\nu_1 \tag{1.34}$$

which corresponds to the limiting value of $f(\nu)$. This term may be transformed into

$$W_1^0 = \frac{2}{\sqrt{\varrho}}e^{i\frac{\zeta^2}{4\varrho}} - q_1\int_0^\zeta \frac{2}{\sqrt{\varrho}}e^{i\frac{\zeta^2}{4\varrho}}\,d\zeta \tag{1.35}$$

W_1^0 satisfies equation (1.25) and the boundary condition (1.26). For $\varrho \to 0$ we have

$$\lim\left(W_1^0 - \frac{2}{\sqrt{\varrho}}e^{i\frac{\zeta^2}{4\varrho}}\right) = -2\sqrt{\pi}\,e^{i\frac{\pi}{4}}q_1 \tag{1.36}$$

for any $\zeta > 0$. Hence if we put

$$W_1 = W_1^0 + W_1' \tag{1.37}$$

the function W_1' has to satisfy equation (1.25) and condition (1.26), while condition (1.27) gives

$$W_1' = 2\sqrt{\pi}\,e^{i\frac{\pi}{4}}q_1 \qquad (\text{for } \varrho = 0, \quad \zeta > 0) \tag{1.38}$$

If we put in (1.32)

$$f(\nu) = \frac{4}{\sqrt{\pi}}e^{i\frac{\pi}{4}}(1 + g(\nu)) \tag{1.39}$$

we get

$$W_1' = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \int_0^\infty e^{-i\nu^2\varrho} \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) g(\nu)\, d\nu + A\, e^{iq_1^2\varrho - q_1\zeta} \tag{1.40}$$

and condition (1.38) becomes

$$\int_0^\infty \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) g(\nu)\, d\nu + \frac{\sqrt{\pi}}{4} A\, e^{-i\frac{\pi}{4}} e^{-q_1\zeta} = \frac{\pi}{2} q_1 \qquad (\text{for } \zeta > 0) \tag{1.41}$$

The exponential function in (1.41) admits an integral representation (valid for $\zeta > 0$)

$$e^{-q_1\zeta} = \frac{2}{\pi} q_1 \int_0^\infty \frac{\cos \nu\zeta}{\nu^2 + q_1^2}\, d\nu \tag{1.42}$$

Multiplying this expression by $q_1\, d\zeta$ and integrating over ζ from 0 to ζ we obtain

$$1 - e^{-q_1\zeta} = \frac{2}{\pi} q_1^2 \int_0^\infty \frac{\sin \nu\zeta}{\nu(\nu^2 + q_1^2)}\, d\nu \tag{1.43}$$

Subtracting (1.42) from (1.43) and multiplying by $\frac{\pi}{2} q_1$ we obtain the equation

$$-q_1^2 \int_0^\infty \left(\cos \nu\zeta - \frac{q_1}{\nu} \sin \nu\zeta\right) \frac{d\nu}{\nu^2 + q_1^2} + \pi q_1\, e^{-q_1\zeta} = \frac{\pi}{2} q_1 \tag{1.44}$$

which is to be compared with (1.41). Identifying (1.44) with (1.41) we obtain

$$g(\nu) = -\frac{q_1^2}{\nu^2 + q_1^2}; \qquad A = 4\sqrt{\pi} \cdot e^{i\frac{\pi}{4}} q_1 \tag{1.45}$$

According to (1.39) it follows

$$f(\nu) = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \frac{\nu^2}{\nu^2 + q_1^2} \tag{1.46}$$

Inserting this and the value (1.45) for A in (1.32), we arrive at the following expression for the function W_1:

$$W_1 = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \left[\int_0^\infty e^{-i\nu^2\varrho} (\nu \cos \nu\zeta - q_1 \sin \nu\zeta) \frac{\nu\, d\nu}{\nu^2 + q_1^2} + \pi q_1\, e^{iq_1^2\varrho - q_1\zeta}\right] \tag{1.47}$$

It is convenient for the investigation of this expression to replace the

integral over the real axis by an integral over the line arc $\nu = -\pi/4$, since the new integral converges more rapidly.

In the sector

$$-\frac{\pi}{4} < \text{arc } \nu < 0 \tag{1.48}$$

between the old and the new integration path there is, however, a pole $\nu = -iq_1$. The residue in this pole exactly cancels the additional term in (1.47), and we obtain

$$W_1 = \frac{4}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \int_0^{\infty e^{-i(\pi/4)}} e^{-i\nu^2\varrho} (\nu \cos \nu\zeta - q_1 \sin \nu\zeta) \frac{\nu \, d\nu}{\nu^2 + q_1^2} \tag{1.49}$$

We can write instead of this

$$W_1 = \frac{2}{\sqrt{\pi}} e^{i\frac{\pi}{4}} \int_{-\infty e^{-i(\pi/4)}}^{+\infty e^{-i(\pi/4)}} e^{-i\nu^2\varrho + i\nu\zeta} \frac{\nu \, d\nu}{\nu - iq_1} \tag{1.50}$$

since the integrand in (1.49) is the even part of the integrand in (1.50). We introduce a new variable of integration p putting

$$\nu = \frac{\zeta}{2\varrho} + \frac{p}{\sqrt{\varrho}} e^{-i\frac{\pi}{4}} \tag{1.51}$$

We can shift the contour to the right a distance $\zeta/2\varrho$ and then the new variable p will be a real quantity running from $-\infty$ to $+\infty$. Putting for brevity

$$e^{-i\frac{\pi}{4}} q_1 \sqrt{\varrho} = \sigma; \qquad e^{i\frac{\pi}{4}} \frac{\zeta}{2\sqrt{\varrho}} = \tau \tag{1.52}$$

we get

$$W_1 = \frac{2}{\sqrt{(\pi\varrho)}} e^{i\frac{\zeta^2}{4\varrho}} \int_{-\infty}^{+\infty} e^{-p^2} \frac{p+\tau}{p+\sigma+\tau} \, dp \tag{1.53}$$

It is convenient now to go from W_1 back to the original "attenuation function" W, according to (1.24). We have

$$W = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2} \frac{p+\tau}{p+\sigma+\tau} \, dp \tag{1.54}$$

This integral can be easily evaluated. It represents different analytic functions according to the sign of the imaginary part of $\sigma+\tau$. But from

(1.30) and (1.52) it follows

$$\operatorname{Im}(\sigma) > 0; \qquad \operatorname{Im}(\tau) > 0 \tag{1.55}$$

so that in our case $\operatorname{Im}(\sigma+\tau) > 0$. In this case the integral (1.54) is equal to

$$W = 2-4\sigma\, \mathrm{e}^{-(\sigma+\tau)^2} \int_{i\infty}^{\sigma+\tau} \mathrm{e}^{\alpha^2}\, d\alpha \tag{1.56}$$

This is the well-known Weyl–van der Pol formula, which we have derived.

As is seen from the derivation, the conditions stated above are sufficient to determine the function W in a unique way. On the contrary, any expression of the form (1.32) (with $f(\nu)$ continuous and absolutely integrable) could be added to the obtained solution without interfering with condition (1.22).

As already pointed out, the necessity of condition (1.23) is connected with the fact that equation (1.08) for W has a singularity at $\varrho=0$ whereas equation (1.25) for W_1 has no singularities.

The derivation of the Weyl–van der Pol formula by the method of parabolic equations is but little easier than the usual derivation. However, in cases more complicated than that of a plane earth, the use of this method leads to great simplifications.

2. *The Case of a Spherical Earth*

Let us denote by r, ϑ, φ spherical coordinates with the origin in the centre of the earth globe and with polar axis drawn through the source (vertical dipole). The electric and the magnetic fields can be expressed by means of the Hertz function as follows:

$$\left.\begin{aligned} E_r &= \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right) \\ E_\vartheta &= -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial\vartheta}\right) \end{aligned}\right\} \tag{2.01}$$

$$H_\varphi = -ik\frac{\partial U}{\partial\vartheta} \tag{2.02}$$

The function U satisfies the differential equation

$$\Delta U+k^2U = 0 \tag{2.03}$$

and also certain boundary conditions on the surface of the globe ($r=a$). As in the plane case we shall consider the modulus of the complex dielectric constant η as a large quantity (compared with unity). This assumption per-

mits us to write the boundary conditions in an approximate form pointed out by M. Leontovich, and repeatedly used for the solution of similar problems ([10], [21]). For the plane case these conditions are of the form (1.09) used above; for the spherical case they become

$$\frac{\partial(rU)}{\partial r} = -\frac{ik}{\sqrt{\eta}} rU \qquad (\text{for } r = a) \tag{2.04}$$

These conditions lead to the following relation for the field components:

$$E_\vartheta = -\frac{1}{\sqrt{\eta}} H_\varphi \qquad (\text{for } r = a) \tag{2.05}$$

The character of the singularity of the Hertz function at the point where the dipole is located is the same as in the plane case. Namely, if the dipole and the point of observation are located above the earth's surface and if R is their mutual distance, then

$$\lim RU = 1 \qquad \text{for} \qquad kR \to 0 \tag{2.06}$$

We shall look for a solution of the form

$$U = \frac{e^{ikR}}{R} W \tag{2.07}$$

where W is the attenuation function. In the following we shall consider the dipole to be located on the earth's surface itself, and, therefore

$$R = \sqrt{(r^2 + a^2 - 2ra \cos \vartheta)} \tag{2.08}$$

Let us examine what are the "small" and "large" parameters, which characterize our problem. First of all, in the case considered, the wave length is extremely small as compared with the radius of the earth. Hence ka is very large as compared with unity (of the order of several millions).

In solving our problem we shall take this circumstance into account from the very beginning; our aim is to find the asymptotic limiting form of the solution for large values of ka. Further, as pointed out above, we consider $|\eta|$ to be large as compared to unity. The ratio of the orders of magnitude of these two large parameters will be examined later. We are concerned with distances which, although large compared with a wave length, are small compared with the radius of the earth.

The idea of our method consists in the following. For large $|\eta|$ and large ka the attenuation function W is a slowly varying function of coordinates, i.e. its relative variation over one wave length is very small. This

is seen, for instance, from the fact that in a very large region $W = 1+f$ where f is the Fresnel coefficient (1.14). To express the slowness of the variation of W in an explicit form we shall introduce large (as compared with the wavelength) scales of lengths: m_r in the direction of the radius vector (in the vertical direction) and m_ϑ in the direction of the meridian arc (in a horizontal direction). Putting

$$r = a+m_r y; \qquad \vartheta = \frac{m_\vartheta}{a}x \tag{2.09}$$

we introduce new dimensionless quantities x, y and assume that

$$W = W(x, y) \tag{2.10}$$

and that derivatives $\partial W/\partial x$ and $\partial W/\partial y$ are of the same order of magnitude as W itself (this expresses the slowness of the variation of W). We shall show that by a suitable choice of the scales m_r and m_ϑ we can (in the case of large ka) obtain for $W(x, y)$ an equation and boundary conditions which do not involve large parameters and which lead to a solution valid in the whole region considered.

Using these assumptions the equation of the plane of the horizon

$$r \cos\vartheta = a \tag{2.11}$$

(the limit of the direct visibility) can be written in the form

$$r = a+a\frac{\vartheta^2}{2} \tag{2.12}$$

or

$$m_r y = \frac{m_\vartheta^2}{2a}x^2 \tag{2.13}$$

From physical considerations it is clear that the boundary of direct visibility must play an essential role in our problem. Therefore, it is convenient to make its equation free from any parameters. This can be done by connecting m_r and m_ϑ by the relation

$$m_r = \frac{m_\vartheta^2}{2a} \tag{2.14}$$

by virtue of which the equation of the boundary of direct visibility assumes the form

$$y = x^2 \tag{2.15}$$

As mentioned above, we look for the solution in the region where $\vartheta \ll \pi/2$. Therefore, we require that values of x of the order of unity should correspond to small values of ϑ. This will be the case if $m_\vartheta \ll a$ or, if we put $m_\vartheta = a/A$ we must consider A as a large number (as compared with unity). Equations (2.01) transform into

$$r = a\left(1+\frac{y}{2A^2}\right); \qquad \vartheta = \frac{x}{A} \tag{2.16}$$

and the distance R from the dipole (formula (2.08)), when expressed in terms of x and y, reduces to

$$R = a\frac{x}{A}\left[1+\frac{1}{4A^2}\left(y+\frac{y^2}{2x^2}-\frac{x^2}{6}\right)\right] \tag{2.17}$$

where in the parentheses the omitted terms are of the order $1/A^4$ and higher.

Let us now derive the approximate differential equation for the attenuation function W. If $\mathbf{R}$ is the radius vector drawn from the dipole, then from (2.03) and (2.07) there follows the equation

$$\Delta W+2\left(ik-\frac{1}{R}\right)\frac{(\mathbf{R}\ \mathrm{grad}\ W)}{R} = 0 \tag{2.18}$$

Transformed to polar coordinates, equation (2.18) takes the form

$$\frac{\partial^2 W}{\partial r^2}+\frac{2}{r}\frac{\partial W}{\partial r}+\frac{1}{r^2}\frac{\partial^2 W}{\partial \vartheta^2}+\frac{\cot\vartheta}{r}\frac{\partial W}{\partial \vartheta}+$$
$$+\frac{2}{R}\left(ik-\frac{1}{R}\right)\left[(r-a\cos\vartheta)\frac{\partial W}{\partial r}+\frac{a}{r}\sin\vartheta\frac{\partial W}{\partial \vartheta}\right] = 0 \tag{2.19}$$

Making a further transformation from the variables r and ϑ to x and y and retaining in the differential equation thus obtained only terms of the highest order in A, we get

$$\frac{\partial^2 W}{\partial y^2}+\frac{ika}{2A^3}\left[\left(x+\frac{y}{x}\right)\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\right] = 0 \tag{2.20}$$

We note that the omitted terms are of the order $1/A^2$ compared with those written down.

As yet we have not fixed the value of the large parameter A. We can try to choose its value in such a manner that for $ka \to \infty$ equation (2.20) does not contain any parameters and that it possesses a solution satisfying the necessary conditions. This is only possible if A^3 is proportional to ka.

Therefore, we put

$$A = \left(\frac{ka}{2}\right)^{1/3}. \tag{2.21}$$

and equation (2.20) takes the form

$$\frac{\partial^2 W}{\partial y^2}+i\left[\left(x+\frac{y}{x}\right)\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x}\right] = 0 \tag{2.22}$$

We note that this equation is simply the equation for the zero-order term $W^{(0)}$ in the expansion

$$W = W^{(0)}+\frac{1}{A^2}W^{(1)}+\dots \tag{2.23}$$

Besides the assumption that A^3 is proportional to ka one could consider two more possibilities. Firstly, we could supppose that

$$\frac{A^3}{ka} \to 0 \qquad \text{for} \qquad ka \to \infty \tag{2.24}$$

or, secondly, that

$$\frac{A^3}{ka} \to \infty \qquad \text{for} \qquad ka \to \infty \tag{2.25}$$

in the first case the limiting form of the equation would be

$$\left(x+\frac{y}{x}\right)\frac{\partial W}{\partial y}+\frac{\partial W}{\partial x} = 0 \tag{2.26}$$

and in the second case

$$\frac{\partial^2 W}{\partial y^2} = 0 \tag{2.27}$$

However, it is easy to prove that the solutions of these equations cannot satisfy the boundary conditions. Thus the only admissible assumption is that made above.

We have now to formulate the boundary conditions. Using (2.17) and (2.21) and retaining only the terms of highest order with respect to A we obtain from (2.04) and (2.07)

$$\frac{\partial(e^{ikR}W)}{\partial y} = -i\frac{A}{\sqrt{\eta}}(e^{ikR}W) \qquad (\text{for } y = 0) \tag{2.28}$$

or to the same approximation

$$\frac{\partial W}{\partial y}+i\left(\frac{A}{\sqrt{\eta}}+\frac{x}{2}\right)W = 0 \qquad (\text{for } y = 0) \tag{2.29}$$

This boundary condition involves the complex quantity

$$q = i\frac{A}{\sqrt{\eta}} = i\left(\frac{\pi a}{\lambda}\right)^{1/3}\frac{1}{\sqrt{\left(\varepsilon+i\frac{\lambda}{2\pi l}\right)}} \tag{2.30}$$

which may be written in the form

$$q = |q|\cdot q_1 \tag{2.31}$$

where the value of q_1 is given by (1.11) and $|q_1|=1$. Since $|q|$ is the ratio of two large parameters, the value of this quantity can be large as well as small.

We introduce a length b (which is independent of the wave length)

$$b = \left(\frac{a}{2}\right)^{2/5} l^{3/5} \tag{2.32}$$

and put

$$n = \frac{2\pi b}{\lambda}; \qquad \alpha = \varepsilon\frac{l}{b} = \varepsilon\left(\frac{2l}{a}\right)^{2/5} \tag{2.33}$$

Then the quantity q can be written in the form

$$\eta = n^{5/6}\frac{i}{\sqrt{(i+\alpha n)}} \tag{2.34}$$

As it is seen from Table 1, the parameter α varies for sea water and for different kinds of soils within relatively narrow limits (approximately from 0·01 to 0·03 and for dry soil to 0·08), whereas the length $2\pi b$ varies from tens to thousands of metres. Therefore, n will be very large (such that $|q|$ is of the order of A^2) only for very short waves and dry soils. In the general case, however, we must consider $|q|$ as finite and retain q in the boundary condition which we shall write in the form

$$\frac{\partial W}{\partial y}+\left(q+\frac{ix}{2}\right)W = 0 \qquad (\text{for } y = 0) \tag{2.35}$$

It is interesting to compare the equations and the boundary conditions for the two cases considered (the case of the plane earth and that of the

TABLE 1

Soil	$\frac{\sigma_0}{\sigma}$	$2\pi l$ (in metres)	$2\pi b$ (in metres)	ε	α
Sea water very salty	2×10^5	0·0032	26·6	80	0·010
Sea water scarcely salty	10^6	0·016	69·8	80	0·018
Ditto	2×10^6	0·032	105	80	0·024
Swamp	10^7	0·16	278	15	0·009
Moist soil	10^8	1·6	1110	15	0·022
and meadows	2×10^8	3·2	1680	15	0·029
Fresh clean water	10^9	16	4420	80	0·29
Dry soil	10^{10}	160	17500	9	0·08

Note. The first column gives the ratio of the conductivity of mercury σ_0 to the conductivity of a given soil σ. The conductivity of mercury is taken to be $\sigma_0=10{\cdot}440\ (\Omega\ \mathrm{cm})^{-1}$.

spherical earth). Putting

$$\varrho = |q|^2 x \qquad \zeta = |q|\, y \tag{2.36}$$

we go back from our variables x, y to the old dimensionless variables ϱ, ζ used in Section 1. Introducing in (2.22) and (2.35) the variables ϱ, ζ we obtain the equations

$$\frac{\partial^2 W}{\partial \zeta^2}+i\left[\frac{\partial W}{\partial \varrho}+\left(\frac{\zeta}{\varrho}+\frac{\varrho}{|q|^3}\right)\frac{\partial W}{\partial \zeta}\right] = 0 \tag{2.37}$$

$$\frac{\partial W}{\partial \zeta}+\left(q_1+\frac{i\varrho}{2|q|^3}\right) W = 0 \ (\text{for } \zeta = 0) \tag{2.38}$$

where the terms of the order $\varrho/|q|^3$ are due to the curvature of the earth. By omitting these terms, we return to equations (1.08) and (1.10) for the plane earth.

We have now to formulate the condition at $x=0$. The corresponding condition for the plane earth has been discussed in Section 1. It has been shown there that we cannot utilize directly the character of the singularity of the Hertz function in the source, but have to consider the region, where the "reflection formula" (1.13) or its limiting form (1.17) is valid and have to compare these formulae with the desired solution in that region.

For the spherical earth the condition at $x=0$ does not differ essentially from the corresponding condition for the plane earth, and we can write it

in the form

$$\left|\frac{W-2}{\sqrt{x}}\right| \to 0 \qquad \text{for} \qquad x \to 0 \qquad \text{and} \qquad y > 0 \tag{2.39}$$

in close analogy to (1.23).

Thus, our problem is to obtain the function W from the differential equation (2.22), conditions (2.35) and (2.39), the condition that W remains finite for all $y>0$, and the "condition at infinity" (p. 216).

The solution of this problem, which is of a purely mathematical nature, can be obtained as follows.

First of all, we simplify the differential equation (2.22) by the substitution

$$W = e^{-i\omega_0} V \tag{2.40}$$

where

$$\omega_0 = \frac{y^2}{4x}+\frac{xy}{2}-\frac{x^3}{12} \tag{2.41}$$

The geometrical interpretation of the quantity ω_0 follows from formula (2.17) which can be written in the form

$$kR = ka\vartheta + \omega_0 \tag{2.42}$$

Thus ω_0 is the difference between the distance R measured along the straight line and the corresponding length of the arc (measured along the earth's surface), both quantities being expressed in wave numbers. According to (2.40) and (2.42) we have

$$e^{ikR}\, W = e^{ika\vartheta}\, V \tag{2.43}$$

so that the transition from W to V corresponds to the separation of the phase factor $e^{ika\vartheta}$ instead of e^{ikR}.

Inserting (2.40) into the differential equation for W and using the relations

$$2\frac{\partial \omega_0}{\partial y} = x+\frac{y}{x}\,; \qquad \left(\frac{\partial \omega_0}{\partial y}\right)^2+\frac{\partial \omega_0}{\partial x} = y; \qquad \frac{\partial^2 \omega_0}{\partial y^2} = \frac{1}{2x} \tag{2.44}$$

we obtain

$$\frac{\partial^2 V}{\partial y^2}+i\frac{\partial V}{\partial x}+\left(y-\frac{i}{2x}\right)V = 0 \tag{2.45}$$

This equation (like the original one) has a singularity at $x=0$, but this

singularity can be removed by the substitution

$$V = \sqrt{x} \cdot W_1. \tag{2.46}$$

The result is

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} + yW_1 = 0 \tag{2.47}$$

The boundary condition for W_1 is the same as for V, namely,

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \qquad (\text{for } y = 0) \tag{2.48}$$

We note that this condition is more simply obtained directly from (2.28) (rather than from (2.35)).

Finally, the condition for $x \to 0$ is

$$W_1 - \frac{2}{\sqrt{x}} e^{i\frac{y^2}{4x}} \to 0 \qquad (\text{for } x \to 0 \quad \text{and} \quad y > 0) \tag{2.49}$$

Transition from W to W_1 simplifies the problem considerably. Firstly, equation (2.47) is not only free from a singularity at $x=0$, but also its coefficients do not contain the argument x; therefore, it is soluble by the method of separation of the variables. Secondly, the coefficient in the boundary condition (2.48) does not involve x. From the fact that $x=0$ is a regular point of equation (2.47), it follows also that condition (2.49) for $x=0$ (together with the other boundary condition) is sufficient for a unique determination of W_1.

We shall solve equation (2.09) by the classical method of separation of the variables. Considering particular solutions of the form

$$W_1 = X(x)Y(y) \tag{2.50}$$

we get the following equations for X and Y

$$\frac{Y''}{Y} + y = -i\frac{X'}{X} = t \tag{2.51}$$

where t is the separation parameter. Hence

$$X' = itX \tag{2.52}$$

$$Y'' + (y-t)Y = 0 \tag{2.53}$$

The solution of equations (2.52) and (2.53) is

$$X(x) = e^{itx} \tag{2.54}$$

$$Y(y) = w(t-y) \tag{2.55}$$

where $w(t)$ is an integral of the equation

$$w''(t) = tw(t) \tag{2.56}$$

For $w(t)$ we may take the function

$$w(t) = \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{tz - {}^1/_3 z^3} \mathrm{d}z \tag{2.57}$$

where the contour Γ is a broken line drawn from infinity to zero along the straight line arc $z = -\frac{2\pi}{3}$ and from zero to infinity along the positive real axis. The function $w(t)$ is an integral transcendental function which can be expressed using the Hankel functions of the first kind and of order 1/3 according to the formula

$$w(t) = e^{i\frac{2\pi}{3}} \sqrt{\frac{\pi}{3}} \cdot (-t)^{1/2} H_{1/3}^{(1)} \left[\frac{2}{3}(-t)^{3/2}\right] \tag{2.58}$$

The properties of $w(t)$ are summarized in Chapter 10. The phase of the function $w(t-y)$ increases as $y \to +\infty$. (The second integral of equation (2.53) which may be written in the form

$$w_2(t-y) = e^{-i\frac{\pi}{3}} w\left[e^{i\frac{2\pi}{3}}(t-y)\right] \tag{2.59}$$

does not possess this property). Expression (2.50) will satisfy the boundary condition (2.48) if we choose the parameter t so as to satisfy the relation

$$w'(t) - qw(t) = 0 \tag{2.60}$$

It was shown in Chapter 10 that all roots t_s of this equation lie in the first quadrant of the t-plane; the distant roots are situated near the straight line arc $t = \pi/3$. Therefrom it follows that the function

$$W_1 = e^{ixt_s} w(t_s - y) \tag{2.61}$$

remains finite for all positive values of x and for finite values of y. (If we took in (2.60) and (2.61) the other solution w_2, then the roots t_s would lie in the fourth quadrant and the exponential e^{ixt_s} would increase indefinitely with increasing x; therefore, the solution with w_2 is to be rejected). Thus the function (2.61) satisfies the differential equation (2.47), the boundary condition (2.48) and remains finite with respect to x (it tends to zero as x increases). All these conditions are also satisfied, for $x > 0$, by the integral

$$W_1 = \int \frac{e^{ixt} w(t-y)}{w'(t) - qw(t)} \psi(t) \, \mathrm{d}t \tag{2.62}$$

where C is a closed contour in the t-plane enclosing the roots of (2.60). In addition, if $\psi(t)$ is holomorphic and bounded inside the contour, the integral remains finite for all values of y, while the particular solution (2.61) is not bounded with respect to y.†

We have now to satisfy equation (2.49). This can be done by a suitable choice of the contour C and of the function $\psi(t)$. It is clear that the contour C must go to infinity, since the integral along any finite contour cannot have a singularity at $x=0$. The singularity is caused by distant parts of the contour. But for large values of $|t|$ the following asymptotic expressions are valid:

$$\frac{w(t-y)}{w'(t)-qw(t)} = \begin{cases} \dfrac{1}{\sqrt{t}}\,e^{-y\sqrt{t}} \left(-\dfrac{2\pi}{3} < \operatorname{arc} t < \dfrac{\pi}{3}\right) \\ -\dfrac{1}{\sqrt{t}}\,e^{y\sqrt{t}} \left(\dfrac{\pi}{3} < \operatorname{arc} t < \dfrac{4\pi}{3}\right) \end{cases} \tag{2.63}$$

where $\operatorname{arc} \sqrt{t} = \dfrac{1}{2} \operatorname{arc} t$ (on the line $\operatorname{arc} t = 4\pi/3$ or $\operatorname{arc} t = 2\pi/3$ the two expressions coincide). The contour C has two branches going to infinity. We shall draw one of them along the positive imaginary axis (from $i\infty$ to 0) and the other along the positive real axis (from 0 to $+\infty$); the lower expression (2.63) is valid on the first branch, the upper on the second branch. The singularity of the integral (2.62) for $x=0$ is the same as that of the integral

$$W_1' = \int_0^{i\infty} e^{ixt+y\sqrt{t}}\,\psi(t)\,\frac{dt}{\sqrt{t}} + \int_0^{\infty} e^{ixt-y\sqrt{t}}\,\psi(t)\,\frac{dt}{\sqrt{t}} \tag{2.64}$$

This is true in spite of the fact that the asymptotic expressions (2.63) are invalid for small and finite values of t, because the integrals over the corresponding parts of the integration path remain finite and have no singularities.

Assuming the function $\psi(t)$ holomorphic and bounded in the first quadrant we can replace the upper limit in the second integral by $i\infty$. Then, putting $t=ip^2$, we get

$$W_1' = 2\,e^{i\frac{\pi}{4}} \int_{-\infty}^{+\infty} e^{-xp^2+\sqrt{i}\cdot yp}\,\psi(ip^2)\,dp \tag{2.65}$$

But we have

$$\int_{-\infty}^{+\infty} e^{-xp^2+\sqrt{i}\cdot yp}\,dp = \sqrt{\frac{\pi}{x}}\cdot e^{i\frac{y^2}{4x}} \tag{2.66}$$

† The text of this paragraph is revised for the present edition.

Therefore, if we suppose that $\psi(t)$ is a constant quantity equal to

$$\psi(t) = \frac{1}{\sqrt{\pi}} e^{-i\frac{\pi}{4}} \tag{2.67}$$

we obtain

$$W_1' = \frac{2}{\sqrt{x}} e^{i\frac{y^2}{4x}} \tag{2.68}$$

which is the required singularity of W_1. Inserting the obtained value of $\psi(t)$ in (2.62) we are led to consider the integral

$$W_1 = \frac{1}{\sqrt{\pi}} e^{-i\frac{\pi}{4}} \int_C \frac{e^{ixt}\, w(t-y)}{w'(t)-qw(t)} \mathrm{d}t \tag{2.69}$$

which satisfies the differential equation and the boundary conditions and has the required singularity for $x=0$. However, we cannot yet assert that the integral (2.69) gives the solution of our problem. In fact, the more general form (2.62) of the integral will have the same singularity, if the function $\psi(t)$ is holomorphic in the first quadrant and tends to a constant value (2.67) at infinity. The more general integral satisfies the following relation

$$\lim_{x\to 0} \left(W_1 - \frac{2}{\sqrt{x}} e^{i\frac{y^2}{4x}} \right) = f(y) \tag{2.70}$$

where $f(y)$ is some bounded function, the form of which depends on $\psi(t)$. But if $\psi(t)$ is a constant, the function $f(y)$ turns out to vanish identically. This can be shown by evaluating the integral (2.69) by the method of stationary phase (the main part of the integration path lies in the neighbourhood of the point $t=-[(y-x^2)/2x]^2$ i.e. for large negative values of t). We shall not perform these calculations since similar ones are made in [10].

Hence expression (2.69) satisfies all conditions including (2.49).

We shall not attempt to give here a rigorous proof of the uniqueness of the solution, but it is clear that by adding expressions of the form (2.61) to the solution obtained, condition (2.49) is violated.

Going back, according to (2.46), to the function V, we get the following expression for this function:

$$V(x, y, q) = e^{-i\frac{\pi}{4}} \sqrt{\frac{x}{\pi}} \int_C \frac{e^{ixt} w(t-y)}{w'(t)-qw(t)} \mathrm{d}t \tag{2.71}$$

Using (2.43) and substituting $a\vartheta$ for R in the denominator of (2.07), we

come to the final expression for the Hertz function

$$U = \frac{e^{ika\vartheta}}{a\vartheta} V(x, y, q) \tag{2.72}$$

This expression coincides exactly with formula (6.01) of Chapter 10 obtained by the method of summation of series.

A detailed discussion of the expression obtained was given in Chapter 10 and will not be repeated here.

Comparing the two methods of derivation of formula (2.71) we arrive at the following conclusions. The method of the summation of series is more cumbersome but it is at the same time more rigorous. This is because all approximations are made in the final solution, which makes an estimation of the order of disregarded terms easier. The method permits condition (2.06) to be used directly without resorting to the "reflection formula" which requires a proof itself. On the other hand, using the method of parabolic equations all neglections are made in the initial equations. This requires careful reasoning which is difficult to perform with complete rigour. The lack of rigour is compensated by the comparative simplicity of the second method. This simplicity is the chief advantage of the method since it allows the possibility of finding approximate solutions of other more difficult problems of the same kind where the exact solution is unknown.

Note† This paper is one of the first attempts to state the diffraction problem in terms of the parabolic equation. The argument used to show the uniqueness of the solution is somewhat incomplete (especially the reason for the choice of $\psi(t)$ in (2.62)),but the results are correct. A simpler and more convincing deduction of the formulae concerned can be obtained if the incident and the reflected wave are considered separately (for the incident wave an expression satisfying the parabolic equation is to be used). This procedure is adopted in our subsequent papers (see Ref. 5). In the text of this chapter only slight changes have been made.

† Added in the present edition.

CHAPTER 12

THE FIELD OF A VERTICAL AND A HORIZONTAL DIPOLE, RAISED ABOVE THE EARTH'S SURFACE†

IN our work on *diffraction of radio waves around the earth's surface* [22] and in Chapter 10 of this Collection we developed a general method for the summation of series representing the field from a dipole above the earth's surface which is supposed to be spherical. In that work our method was applied to the case of a vertical dipole, located on the surface itself. The case of a slightly raised vertical dipole, which is of no less interest, will be investigated in the present paper.

1. *Vertical Raised Dipole. Solution in Series Form*

We will adopt the notation used in Chapter 10. Let k be the wave vector in air, η the complex dielectric constant of the earth, and $k_2=k\eta^{1/2}$ the complex wave vector for the earth. For the sake of simplicity we will not consider the atmospheric refraction; the refraction effect can be approximately taken into account, if we replace the earth's geometrical radius a by the "equivalent" radius a^* (see Chapter 13).

We introduce spherical coordinates r, ϑ, φ with the origin at the centre of the earth and with the polar axis going through the dipole. The field components in air are expressed in terms of the Hertz function U by the formulae

$$E_r = \frac{1}{r\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right) \tag{1.01}$$

$$E_\vartheta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) \tag{1.02}$$

$$H_\varphi = -ik\frac{\partial U}{\partial\vartheta} \tag{1.03}$$

† Fock, 1949.

Let the elevation of the dipole above the earth's surface be $h=b-a$ (so that b is the distance of the dipole from the centre of the earth). We introduce the functions $\psi_n(x)$ and $\zeta_n(x)$, related to the Bessel and the Hankel functions as follows:

$$\psi_n(x) = \sqrt{\left(\frac{\pi x}{2}\right)}\, J_{n+1/2}(x) \tag{1.04}$$

$$\zeta_n(x) = \sqrt{\left(\frac{\pi x}{2}\right)}\, H^{(1)}_{n+1/2}(x). \tag{1.05}$$

We denote by $\chi_n(x)$ the logarithmic derivative

$$\chi_n(x) = \frac{\psi_n'(x)}{\psi_n(x)} \tag{1.06}$$

and by $P_n(\cos\vartheta)$ the Legendre polynomial.

Using these notations we can write the expansion of the Hertz function U in the region $a \leqslant r < b$ in the form

$$U = \frac{i}{kbr}\sum_{n=0}^{\infty}(2n+1)\zeta_n(kb)\left[\psi_n(kr)-A_n\zeta_n(kr)\right]P_n(\cos\vartheta) \tag{1.07}$$

where

$$A_n = \frac{k_2\psi_n'(ka)-k\psi_n(ka)\chi_n(k_2a)}{k_2\zeta_n'(ka)-k\zeta_n(ka)\chi_n(k_2a)} \tag{1.08}$$

These formulae are given in Ref. 22 where, however, as well as in Chapter 10, the calculations are carried out only for the case $r=a$. We shall now rid ourselves of this limitation.

2. *Approximate Summation of the Series for the Hertz Function*

For the approximate summation of the series we can apply unchanged the method exposed in Chapter 10. We write the series (1.07) in the form

$$U = \sum_{n=0}^{\infty}(n+{}^1/_2)\varphi(n+{}^1/_2)P_n(\cos\vartheta) \tag{2.01}$$

where

$$\varphi(n+{}^1/_2) = \frac{2i}{kbr}\zeta_n(kb)\left[\psi_n(kr)-A_n\zeta_n(kr)\right] \tag{2.02}$$

We put $n+\frac{1}{2}=\nu$ and consider ν as a complex variable. The function $\varphi(\nu)$ is an analytic function of ν with poles only in the first quadrant. As

shown in Section 2 of Chapter 10, on condition $ka \gg 1$ the sum (2.01) can be replaced to a good approximation by the integral

$$U = \frac{e^{-i\pi/4}}{\sqrt{(2\pi \sin \vartheta)}} \int_C e^{i\nu\vartheta} \varphi(\nu) \sqrt{\nu} \cdot d\nu \tag{2.03}$$

where the contour C runs from infinity in the second quadrant, encloses all the poles of $\varphi(\nu)$ and recedes to infinity in the first quadrant of the plane of the complex variable ν.

The principal segment of the contour will be that on which

$$\nu = ka + \left(\frac{ka}{2}\right)^{1/3} t \tag{2.04}$$

where $|t|$ is bounded (while ka is supposed to be very large). The quantity

$$m = \left(\frac{ka}{2}\right)^{1/3} \tag{2.05}$$

represents the "large parameter" of our problem (terms of the order of $1/m^2$ in comparison with unity will be neglected). This quantity will frequently appear in further calculations.

On the relevant segment of the integration path we can replace the functions $\psi_n(x)$ and $\zeta_n(x)$ by their asymptotic expressions in terms of the Airy functions, investigated in detail in Ref. 22 (See Appendix). We shall use four Airy functions: $u(t)$, $v(t)$, $w_1(t)$ and $w_2(t)$. These functions represent solutions of the differential equation

$$w''(t) = tw(t) \tag{2.06}$$

connected by the relations

$$w_1(t) = u(t) + iv(t); \qquad w_2(t) = u(t) - iv(t) \tag{2.07}$$

For real t the functions $u(t)$ and $v(t)$ are real. The function $w_1(t)$ is expressible in terms of the Hankel function of the first kind and of order 1/3. We have

$$w_1(t) = e^{\frac{2}{3}\pi i} \sqrt{\left(\frac{\pi}{3}\right)} \cdot (-t)^{1/2} H^{(1)}_{1/3}[{}^2/_3(-t)^{3/2}] \tag{2.08}$$

Sometimes we will drop the suffix 1 in $w_1(t)$ and write simply $w(t)$.

The asymptotic expressions for the function ζ_n and its derivative have the form

$$\zeta_n(ka) = -im^{1/2} w_1(t); \qquad \zeta'(ka) = im^{-1/2} w_1'(t) \tag{2.09}$$

Taking the real parts, we obtain

$$\psi_n(ka) = m^{1/2}v(t); \qquad \psi_n'(ka) = -m^{-1/2}v'(t) \tag{2.10}$$

According to formula (3.08) in Chapter 10, the quantity $\chi_n(ka)$ can be replaced, for $\nu = ka$, by the expression

$$\chi_n(k_2a) = -i\sqrt{\left(1-\frac{k^2}{k_2^2}\right)} = -i\sqrt{\left(\frac{\eta-1}{\eta}\right)} \tag{2.11}$$

Putting

$$q = im\frac{\sqrt{(\eta-1)}}{\eta} \sim \frac{im}{\sqrt{(\eta+1)}} \tag{2.12}$$

and substituting the above expressions in the formula (1.08) for A_n we obtain

$$A_n = i\frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)} \tag{2.13}$$

The expression (2.02) for $\varphi(\nu)$ involves also functions $\psi_n(kr)$, $\zeta_n(kr)$ and $\zeta_n(kb)$. Their asymptotic expressions may be easily obtained from the preceding formulae. We set

$$y_1 = \frac{k}{m}(r-a) = \frac{kh_1}{m} \tag{2.14}$$

$$y_2 = \frac{k}{m}(b-a) = \frac{kh_2}{m} \tag{2.15}$$

where h_2 is the height of the source, h_1 is the height of the observation point and y_2 and y_1 are the corresponding "reduced" heights. We have

$$\zeta_n(kb) = -im^{1/2}w_1(t-y_2) \tag{2.16}$$

$$\zeta_n(kr) = -im^{1/2}w_1(t-y_1) \tag{2.17}$$

$$\psi_n(kr) = m^{1/2}v(t-y_1) \tag{2.18}$$

and the function $\varphi(\nu)$ assumes the form

$$\varphi(\nu) = \frac{1}{am^2}F(t, y_1, y_2, q) \tag{2.19}$$

where

$$F = w_1(t-y_2)\left[v(t-y_1) - \frac{v'(t)-qv(t)}{w_1'(t)-qw_1(t)}w_1(t-y_1)\right] \tag{2.20}$$

Expressing v in terms of w_1 and w_2 we can also write

$$F = \frac{i}{2} w_1(t-y_2)\left[w_2(t-y_1) - \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} w_1(t-y_1)\right] \quad (2.21)$$

The expression we have found for $\varphi(\nu)$ is to be substituted into the integral (2.03) for the Hertz function. Introducing the horizontal distance $s=a\vartheta$, measured along the arc on the earth's surface, and the "reduced horizontal distance"

$$x = \left(\frac{k}{2a^2}\right)^{1/3} s = \frac{ms}{a} \quad (2.22)$$

and replacing $\nu^{1/2}$ in (2.03) by the constant $(ka)^{1/2}$, we obtain

$$U = \frac{e^{iks}}{\sqrt{(sa \sin (s/a))}} V(x, y_1, y_2, q) \quad (2.23)$$

where

$$V = e^{-i\pi/4} \sqrt{\frac{x}{\pi}} \cdot \int_C e^{ixt} F(t, y_1, y_2;\ q)\, dt \quad (2.24)$$

This formula is valid for $y_1 < y_2$; if on the contrary $y_1 > y_2$ the quantities y_1 and y_2 in the expression for F are to be interchanged. The function V is called the attenuation factor.

3. *The Attenuation Factor*

Turning now to the study of the attenuation factor, we will at first examine some limiting cases. We put $y_1=0$, which corresponds to the case when one of the points (source or observation point) is situated on the surface of the earth. We then obtain

$$F(t, 0, y_2, q) = \frac{w_1(t-y_2)}{w_1'(t)-qw_1(t)} \quad (3.01)$$

and formula (2.24) for the attenuation factor reduces to formula (6.02) in Chapter 10.

We next consider the case, when x and y_2 are very large, but the difference

$$x - \sqrt{y_2} = \xi \quad (3.02)$$

is finite. Replacing in (2.21) the function $w_1(t-y_2)$ by its asymptotic

expression [equation (6.18) on p. 211] valid for large y_2, we have

$$V(x, y_1, y_2, q) = \left(\frac{x^2}{y_2}\right)^{1/4} e^{i\frac{2}{3}y_2^{3/2}} \times$$
$$\times \frac{i}{2\sqrt{\pi}} \int e^{i\xi t} \left\{ w_2(t-y_1) - \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} w_1(t-y_1) \right\} dt \tag{3.03}$$

Here the integral coincides with the expression obtained in Chapter 5 (equation (4.39)), as it ought to do. Indeed, for large x and y_2, i.e. for large distances of the source from the point of observation and from the surface of the earth, a wave, proceeding from the source, may be regarded as plane.

In the general case the integral (2.24) for V may be evaluated as a sum of residues. The function F, defined by (2.20) and (2.21), may be written in the form

$$F = w_1(t-y_2) \frac{f(y_1, t)}{w_1'(t)-qw_1(t)} \tag{3.04}$$

where

$$f(y_1, t) = [w_1'(t)-qw_1(t)]v(t-y_1) - [v'(t)-qv(t)]w_1(t-y_1) \tag{3.05}$$

We note that for $y_1=0$ the function f and its derivative $\partial f/\partial y_1$ take the values

$$f = 1; \qquad \frac{\partial f}{\partial y_1} = -q \qquad \text{for} \qquad y_1 = 0 \tag{3.06}$$

Hence it is easy to see that if t is a root of the equation

$$w_1'(t) - qw_1(t) = 0 \qquad t = t_1, t_2, \ldots, \tag{3.07}$$

then the value of the function f coincides with the value of the expression

$$f(y_1, t_s) = f_s(y_1) = \frac{w_1(t_s-y_1)}{w_1(t_s)} \tag{3.08}$$

which may be called the "height factor".

Evaluating the integral (2.24) as the sum of residues at the poles $t=t_s$, we obtain for the attenuation factor V the following expression:

$$V(x, y_1, y_2, q) = e^{i\frac{\pi}{4}} 2\sqrt{(\pi x)} \cdot \sum_{s=1}^{\infty} \frac{e^{ixt_s}}{t_s-q^2} \frac{w_1(t_s-y_1)}{w_1(t_s)} \frac{w_1(t_s-y_2)}{w_1(t_s)} \tag{3.09}$$

This differs from our previous expression for V, corresponding to the case $y_1=0$ (see equation (6.10) in Chapter 10) only in that now two height factors are involved in each term instead of one. The elevations y_1 and y_2 enter quite symmetrically in (3.09).

Using the relation (3.07) we can write equation (3.09) for V in the form

$$V = -e^{i\frac{1\pi}{4}} 2\sqrt{(\pi x)} \cdot \sum_{s=1}^{\infty} \frac{e^{ixt_s}}{1-t_s/q^2} \frac{w_1(t_s-y_1)}{w_1'(t_s)} \frac{w_1(t_s-y_2)}{w_1'(t_s)} \tag{3.10}$$

This form is convenient for calculations if q is large. In particular, for $q=\infty$ we have

$$V = -e^{i\pi/4} 2\sqrt{(\pi x)} \sum_{s=1}^{\infty} e^{ixt_s^0} \frac{w_1(t_s^0-y_1)}{w_1'(t_s^0)} \frac{w_1(t_s^0-y_2)}{w_1'(t_s^0)} \tag{3.11}$$

where the quantities t_s^0 are roots of the equation

$$w_1(t_s^0) = 0 \tag{3.12}$$

The series thus obtained are suitable for calculations in the region of geometrical shadow. In the illuminated region they converge very slowly, but there the reflection formula, which will be developed in the next section, may be used. For the calculation of V in the penumbra region one must resort to numerical integration.

4. *The Reflection Formula*

We now consider the field in the illuminated region. It is to be expected that in this case a reflection formula will be obtained which corresponds to the reflection of spherical waves from a spherical surface.

In the integral (2.24) we may take for F the expression (2.21). This expression contains two terms. The integrals from each of these terms separately may not converge (only their difference being convergent) but, applying the method of stationary phase, we can confine ourselves to the consideration of those segments of the integration path which lie near an extremum of the phase, and may then examine each integral (taken along the respective segment) separately.

We put

$$V^0 = \frac{1}{2} e^{i\pi/4} \sqrt{\left(\frac{x}{\pi}\right)} \int e^{ixt} w_1(t-y_2) w_2(t-y_1)\, dt \tag{4.01}$$

$$V^* = \frac{1}{2} e^{i\pi/4} \sqrt{\left(\frac{x}{\pi}\right)} \int e^{ixt} \frac{w_2'(t)-qw_2(t)}{w_1'(t)-qw_1(t)} w_1(t-y_1) w_1(t-y_2)\, dt \tag{4.02}$$

Then the attenuation factor V will be equal to the difference

$$V = V^0 - V^* \tag{4.03}$$

Supposing that the relevant segment of the integration path lies in the region of large negative values of t and crosses the negative real t-axis from the left downwards to the right, we can replace the functions w_1 and w_2 by their asymptotic expressions valid in this region. According to equation (6.17) in Chapter 10, we may put

$$w_1(t-y) = e^{i\pi/4}(y-t)^{-1/4}e^{i\frac{2}{3}(y-t)^{3/2}} \tag{4.04}$$

and similarly

$$w_2(t-y) = e^{-i\pi/4}(y-t)^{-1/4}e^{-i\frac{2}{3}(y-t)^{3/2}} \tag{4.05}$$

Inserting (4.04) and (4.05) in (4.01) we obtain

$$V^0 = \frac{1}{2}e^{i\pi/4}\sqrt{\left(\frac{x}{\pi}\right)}\int e^{i\omega(t)}\frac{dt}{[(y_1-t)(y_2-t)]^{1/4}} \tag{4.06}$$

where

$$\omega(t) = xt+\frac{2}{3}(y_2-t)^{3/2}-\frac{2}{3}(y_1-t)^{3/2} \tag{4.07}$$

Determining t from the condition $\omega'(t)=0$ we have

$$\sqrt{(y_1-t)} = \frac{y_2-y_1-x^2}{2x}\,; \qquad \sqrt{(y_2-t)} = \frac{y_2-y_1+x^2}{2x} \tag{4.08}$$

For the applicability of expressions (4.04) and (4.05), both quantities (4.08) must be large compared with unity. The extremum value of $\omega(t)$ we denote by ω. This quantity is equal to

$$\omega = \frac{(y_1-y_2)^2}{4x}+\frac{1}{2}x(y_1+y_2)-\frac{1}{12}x^3 \tag{4.09}$$

Application of the method of stationary phase to the integral (4.06) gives

$$V^0 = e^{i\omega} \tag{4.10}$$

The quantity ω has a simple geometrical meaning, namely

$$\omega = k(R-s) \tag{4.11}$$

where R is the distance between the source and the observation point, measured along a straight line, and s is the corresponding horizontal distance, measured along the arc of the earth's circumference. From this it is clear that the quantity V_0 corresponds to the incident wave.

We now examine the integral V^*. Inserting in (4.02) the asymptotic expressions (4.04) and (4.05), we obtain

$$V^* = \frac{1}{2} e^{i\pi/4} \sqrt{\left(\frac{x}{\pi}\right)} \int e^{i\varphi(t)} \frac{q-i\sqrt{-t}}{q+i\sqrt{-t}} \frac{dt}{[(y_1-t)(y_2-t)]^{1/4}} \tag{4.12}$$

where

$$\varphi(t) = xt + \frac{2}{3}(y_1-t)^{3/2} + \frac{2}{3}(y_2-t)^{3/2} - \frac{4}{3}(-t)^{3/2} \tag{4.13}$$

The root of the equation $\varphi'(t)=0$ will be denoted by $t=-p^2$ where $p>0$. The quantity p will satisfy the equation

$$\sqrt{(y_1+p^2)} + \sqrt{(y_2+p^2)} = 2p + x \tag{4.14}$$

which is reducible to a cubic equation. The value of the phase $\varphi(t)$ at $t=-p^2$ will be denoted by φ. Using the relation (4.14) we can eliminate in the expression for $\varphi(t)$ all the radicals except p and write φ in the form

$$\varphi = -3p^2x + 2p(y_1+y_2-x^2) + x(y_1+y_2) - \frac{1}{3}x^3 \tag{4.15}$$

Evaluation of the integral V^* by the method of stationary phase gives

$$V^* = \frac{q-ip}{q+ip} \sqrt{A} \cdot e^{i\varphi} \tag{4.16}$$

where

$$A = \frac{px}{3px + x^2 - y_1 - y_2} \tag{4.17}$$

The formula obtained has a simple geometrical interpretation. The quantity p is equal to

$$p = m \cos\gamma = \left(\frac{ka}{2}\right)^{\frac{1}{3}} \cos\gamma \tag{4.18}$$

where γ is the angle of incidence of the ray (Fig. 1).

The factor $\dfrac{q-ip}{q+ip}$ is the Fresnel coefficient (with its sign reversed) The quantity $\sqrt{A}$ is the correction for the broadening of the bundle of rays after reflection, multiplied by R/r. The phase φ is approximately given by

$$\varphi = k(r_1+r_2-s) \tag{4.19}$$

where r_1, and r_2 are the paths the beam travelled before and after reflection. The expression for the integral V obtained by the method of stationary phase, namely

$$V = e^{i\omega} - \frac{q-ip}{q+ip}\sqrt{A}\cdot e^{i\varphi} \tag{4.20}$$

agrees exactly with the reflection formula. It must be emphasized that this expression (and consequently the reflection formula also) is valid only under the condition that the quantity p be large compared with unity (in practice it is sufficient to require that $p > 2$, or better, $p > 3$).

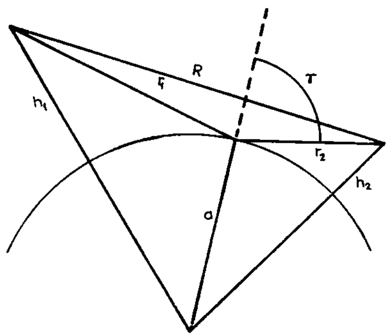

FIG. 1.

If x, y_1 and y_2 are prescribed then the quantity p is determined from equation (4.14). The simplest way to solve this equation is the following. We introduce a new unknown quantity z, putting

$$\sqrt{(y_1+p^2)}-p = \frac{1}{2}(x+z) \tag{4.21}$$

$$\sqrt{(y_2+p^2)}-p = \frac{1}{2}(x-z) \tag{4.22}$$

Solving equation (4.21) for p, we have

$$p = \frac{y_1}{x+z} - \frac{1}{4}(x+z) \tag{4.23}$$

while (4.22) gives

$$p = \frac{y_2}{x-z} - \frac{1}{4}(x-z) \tag{4.24}$$

Equating the two expressions for p, we obtain a cubic equation

$$z^3 - z(x^2 + 2y_1 + 2y_2) + 2x(y_1 - y_2) = 0 \tag{4.25}$$

which is not difficult to solve. We set

$$\varrho^2 = \frac{1}{3}(x^2 + 2y_1 + 2y_2) \qquad (\varrho > 0) \tag{4.26}$$

$$\sin\alpha = \frac{x(y_1 - y_2)}{\varrho^3} \qquad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right) \tag{4.27}$$

Then the relevant root of the cubic equation (4.25) is

$$z = 2\varrho \sin\left(\frac{\alpha}{3}\right) \tag{4.28}$$

Using the cubic equation, one may express p in terms of z in the form

$$2px = y_1 + y_2 - \frac{1}{2}(x^2 + z^2) \tag{4.29}$$

The expression for V simplifies if we introduce in addition to p the quantity

$$p_1 = \frac{x^2 - z^2}{2x} = \frac{2px + x^2 - y_1 - y_2}{x} \tag{4.30}$$

We will then have

$$V = e^{i\omega}\left(1 - \frac{q - ip}{q + ip}\sqrt{\left(\frac{p}{p + p_1}\right)} \cdot e^{2ip_1p^2}\right). \tag{4.31}$$

We observe that the quantities x, z, and p_1 are approximately equal to

$$x = \left(\frac{k}{2a^2}\right)^{1/3}(r_1 + r_2); \qquad z = \left(\frac{k}{2a^2}\right)^{1/3}(r_1 - r_2)$$

$$p_1 = \left(\frac{k}{2a^2}\right)^{1/3}\frac{2r_1r_2}{(r_1 + r_2)} \tag{4.32}$$

If the height y_1 is zero we have $y_1 = 0$ and

$$z = -x; \qquad p = \frac{(y_2 - x^2)}{2x}; \qquad p_1 = 0 \tag{4.33}$$

If the heights are equal, then, putting $y_1 = y_2 = y$ we get

$$z = 0; \qquad p = \frac{y}{x} - \frac{x}{4}; \qquad p_1 = \frac{x}{2} \tag{4.34}$$

We now set $x-\sqrt{y_2}=\xi$ as in (3.02) and increase x and $\sqrt{y_2}$, keeping ξ finite. This corresponds to the transition to a plane incident wave. Putting for brevity

$$\sqrt{(\xi^2+3y_1)} = \sigma \tag{4.35}$$

we have

$$z = -x+\frac{2}{3}(\sigma+\xi); \qquad p = \frac{1}{3}(\sigma-2\xi); \qquad p_1 = \frac{2}{3}(\sigma+\xi) \tag{4.36}$$

Inserting these values of p and p_1 in equation (4.31) we obtain for the integral in (3.03) an expression which coincides with that obtained in Section 6 of Chapter 5 for the case of a plane wave.

5. *Horizontal Electric Dipole. Primary Field*

The field of an electric dipole may be written in the form

$$\mathbf{E} = \operatorname{grad}\operatorname{div}\Pi-\Delta\Pi; \qquad \mathbf{H} = -ik\operatorname{curl}\Pi \tag{5.01}$$

where Π is the Hertz vector, directed along the axis of the dipole and proportional to the quantity

$$\Pi_0 = e^{ikR}/R \tag{5.02}$$

where

$$R = \sqrt{(b^2+r^2-2br\cos\vartheta)} \tag{5.03}$$

Our spherical coordinates are connected with the Cartesian coordinates by the relations

$$x = r\sin\vartheta\cos\varphi; \qquad y = r\sin\vartheta\sin\varphi; \qquad z = r\cos\vartheta \tag{5.04}$$

Supposing the dipole is located at the point $x=0$, $y=0$, $z=b$ and is directed along the x-axis, we can write

$$\Pi_x = \Pi_0; \qquad \Pi_y = 0; \qquad \Pi_z = 0 \tag{5.05}$$

The field components are obtained after inserting the vector (5.05) into the formula (5.01). In the following we shall need only the radial components of the primary dipole field or the quantities rE_r^0 and rH_r^0 proportional to them. Because of the conditions

$$\operatorname{div}\mathbf{E} = 0; \qquad \operatorname{div}\mathbf{H} = 0 \tag{5.06}$$

the latter quantities are solutions of the scalar wave equation. Using

(5.01) and (5.05) we obtain for rE_r^0 the following expression

$$rE^0 = \cos\varphi \frac{\partial}{\partial\vartheta}\left(\cos\vartheta\frac{\partial \Pi_0}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial \Pi_0}{\partial\vartheta}\right) - \cos\varphi\left(\sin\vartheta\frac{\partial \Pi_0}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial \Pi}{\partial\vartheta}\right) \tag{5.07}$$

Since Π_0 depends on r, ϑ, and b only through the quantity R, we have

$$\cos\vartheta\frac{\partial \Pi_0}{\partial r} - \frac{\sin\vartheta}{r}\frac{\partial \Pi_0}{\partial\vartheta} = -\frac{\partial \Pi_0}{\partial b} \tag{5.08}$$

$$\sin\vartheta\frac{\partial \Pi_0}{\partial r} + \frac{\cos\vartheta}{r}\frac{\partial \Pi_0}{\partial\vartheta} = \frac{1}{b}\frac{\partial \Pi_0}{\partial\vartheta} \tag{5.09}$$

Therefore we can write instead of (5.07)

$$rE_r^0 = -\cos\varphi\frac{\partial}{\partial\vartheta}\left(\frac{\partial \Pi_0}{\partial b} + \frac{\Pi_0}{b}\right) \tag{5.10}$$

The quantity rH_r^0 is immediately obtained from (5.01) in the form

$$rH_r^0 = ik\sin\varphi\frac{\partial \Pi_0}{\partial\vartheta} \tag{5.11}$$

In our equations we used the superscript 0 in E_r and H_r to emphasize the fact that these quantities refer to the primary field.

On the other hand, the total field can be expressed by means of two auxiliary functions u and v according to the formulae

$$\begin{aligned} E_r &= \frac{1}{r}\Delta^* u \\ E_\vartheta &= -\frac{1}{r}\frac{\partial^2(ru)}{\partial r\,\partial\vartheta} + \frac{i\omega}{c\sin\vartheta}\frac{\partial v}{\partial\varphi} \\ E_\varphi &= -\frac{1}{r\sin\vartheta}\frac{\partial^2(ru)}{\partial r\,\partial\varphi} - i\frac{\omega}{c}\frac{\partial v}{\partial\vartheta} \end{aligned} \tag{5.12}$$

$$\begin{aligned} H_r &= -\frac{1}{r}\Delta^* v \\ H_\vartheta &= \frac{ick^2}{\omega\sin\vartheta}\frac{\partial u}{\partial\varphi} + \frac{1}{r}\frac{\partial^2(rv)}{\partial r\partial\vartheta} \\ H_\varphi &= -i\frac{ck^2}{\omega}\frac{\partial u}{\partial\vartheta} + \frac{1}{r\sin\vartheta}\frac{\partial^2(rv)}{\partial r\,\partial\varphi} \end{aligned} \tag{5.13}$$

where Δ^* is the Laplace operator on the sphere

$$\Delta^* u = \frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial v}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} \tag{5.14}$$

The functions u and v may be interpreted as the electric and magnetic Hertz functions; sometimes they are called Debye potentials. These functions satisfy the scalar wave equation.

The equations (5.12) give both the field in air and the field in the earth. For air we must put $k=\omega/c$ and for the earth $k=k_2=\frac{\omega}{c}\sqrt{\eta}$ where ω is the frequency; in the following we will mean by k the value of this quantity for air.

The boundary conditions for the electric and magnetic Hertz function follow from the continuity of the tangential components of the field on the earth's surface. Denoting by the superscript (2) those quantities which refer to the field in the earth, we can write the boundary conditions in the form

$$k^2 u = k_2^2 u^{(2)}; \quad \frac{\partial(ru)}{\partial r} = \frac{\partial(ru^{(2)})}{\partial r}; \qquad (\text{for } r=a) \tag{5.15}$$

$$v = v^{(2)}; \quad \frac{\partial(rv)}{\partial r} = \frac{\partial(rv^{(2)})}{\partial r}; \qquad (\text{for } r=a) \tag{5.16}$$

We must find the functions $u=u^0$ and $v=v^0$ corresponding to the primary field in air. Equating the expressions for rE_r^0 which follow from (5.10) and (5.12), we obtain

$$\Delta^* u^0 = -\cos\varphi\frac{\partial}{\partial\vartheta}\left(\frac{\partial\Pi_0}{\partial b}+\frac{\Pi_0}{b}\right) \tag{5.17}$$

and similarly

$$\Delta^* v^0 = -ik\sin\varphi\frac{\partial\Pi_0}{\partial\vartheta} \tag{5.18}$$

From these relations it is easy to determine u^0 and v^0 if we write Π_0 in the form of a series

$$\Pi_0 = \frac{e^{ikR}}{R} = \frac{i}{kbr}\sum_{n=0}^{\infty}(2n+1)\zeta_n(kb)\psi_n(kr)P_n(\cos\vartheta) \tag{5.19}$$

valid for $r<b$. The results are more conveniently written in the form

$$u^0 = -\cos\varphi\frac{\partial P^0}{\partial\vartheta}; \qquad v^0 = -\sin\varphi\frac{\partial Q^0}{\partial\vartheta} \tag{5.20}$$

where P^0 and Q^0 are new auxiliary functions which also satisfy the scalar wave equation, but do not depend on the angle φ. We have

$$P^0 = -\frac{i}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n'(kb)\psi_n(kr)\cdot P_n(\cos\vartheta) \tag{5.21}$$

$$Q^0 = \frac{1}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n(kb)\psi_n(kr)P_n(\cos\vartheta) \tag{5.22}$$

6. *Series for the Total Field*

We represent the functions u and v, in terms of which, according to (5.12) and (5.13), the total field is expressed, in the form

$$u = -\cos\varphi\frac{\partial P}{\partial\vartheta}; \qquad v = -\sin\varphi\frac{\partial Q}{\partial\vartheta} \tag{6.01}$$

where P and Q satisfy the scalar wave equation and the boundary conditions, arising from (5.15) and (5.16):

$$k^2P = k_2^2P^{(2)}; \quad \frac{\partial(rP)}{\partial r} = \frac{\partial(rP^{(2)})}{\partial r} \qquad (\text{for } r = a) \tag{6.02}$$

$$Q = Q^{(2)}; \quad \frac{\partial(rQ)}{\partial r} = \frac{\partial(rQ^{(2)})}{\partial r} \qquad (\text{for } r = a) \tag{6.03}$$

The functions P and Q do not depend on the angle φ. Keeping in mind the form of the primary excitation (5.21), we can write a series for P in air and in the earth as follows

$$P = -\frac{i}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n'(kb)\left[\psi_n(kr)-A_n\zeta_n(kr)\right]P_n(\cos\vartheta) \qquad (a<r<b) \tag{6.04}$$

$$P^{(2)} = -\frac{i}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n'(kb)A_n'\psi_n(k_2r)P_n(\cos\vartheta) \qquad (0<r<a) \tag{6.05}$$

The boundary conditions (6.02) give for the coefficients A_n and A_n' the equations

$$\left.\begin{aligned} k^2A_n\zeta_n(ka)+k_2^2A_n'\psi_n(k_2a) &= k^2\psi_n(ka) \\ kA_n\zeta_n'(ka)+k_2A_n'\psi_n'(k_2a) &= k\psi'(ka) \end{aligned}\right\} \tag{6.06}$$

from which

$$A_n = \frac{k_2\psi_n'(ka)\psi_n(k_2a)-k\psi_n(ka)\psi_n'(k_2a)}{k_2\zeta_n'(ka)\psi_n(k_2a)-k\zeta_n(ka)\psi'(k_2a)} \tag{6.07}$$

$$A_n' = \frac{ik^2/k_2}{k_2\zeta_n'(ka)\psi_n(k_2a)-k\zeta_n(ka)\psi_n'(k_2a)} \tag{6.08}$$

We note that the coefficient A_n here is exactly the same as in the series (1.07) for the Hertz function U of a vertical dipole. Comparison of the series (1.07) and (6.04) for U and for P shows that these functions are connected by the relation

$$\Delta^* P = \frac{\partial U}{\partial b} + \frac{U}{b} - \left(\frac{\partial U}{\partial b} + \frac{U}{b}\right)^0 \tag{6.09}$$

where $(\partial U/\partial b + U/b)^0$ is the constant term in the expansion of the quantity in brackets in a series of Legendre polynomials. This connection between P and U permits us to use in the following the results already obtained for the series summation and to express P in terms of the known attenuation factor V.

In a similar manner we can obtain series for the function Q in air and in the earth. Remembering (5.22), we can write

$$Q = \frac{1}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n(kb)\left[\psi_n(kr) - B_n\zeta_n(kr)\right]P_n(\cos\vartheta) \qquad (a < r < b) \tag{6.10}$$

$$Q^{(2)} = \frac{1}{br}\sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}\zeta_n(kb)B_n'\psi_n(k_2 r)P_n(\cos\vartheta) \qquad (0 < r < a) \tag{6.11}$$

The boundary conditions (6.03) give

$$\begin{aligned} B_n\zeta_n(ka) + B_n'\psi_n(k_2 a) &\equiv \psi_n(ka) \\ kB_n\zeta_n'(ka) + k_2 B_n'\psi_n'(k_2 a) &= k\psi_n'(ka) \end{aligned} \tag{6.12}$$

from which

$$B_n = \frac{k\psi_n'(ka)\psi_n(k_2 a) - k_2\psi_n(ka)\psi_n'(k_2 a)}{k\zeta_n'(ka)\psi_n(k_2 a) - k_2\zeta_n(ka)\psi_n'(k_2 a)} \tag{6.13}$$

$$B_n' = \frac{ik}{k\zeta_n'(ka)\psi_n(k_2 a) - k_2\zeta_n(ka)\psi_n'(k_2 a)} \tag{6.14}$$

The series for the function Q are thus determined. Q is connected with the Hertz function for a vertical magnetic dipole (horizontal loop antenna). The field from such a dipole can be represented by the formulae (5.12) and (5.13), with

$$u = 0; \qquad v = (M/b)W \tag{6.15}$$

where the factor M is the magnetic moment, and the function W is has the same singularity as Π_0 and satisfies the same boundary conditions (5.16) as v.

For the function W in air we obtain an expansion of the form

$$W = \frac{i}{kbr} \sum_{n=0}^{\infty} (2n+1)\zeta_n(kb)\left[\psi_n(kr) - B_n\zeta_n(kr)\right]P_n(\cos\vartheta) \qquad (6.16)$$

where B_n is given by (6.13). Comparison of the series (6.10) and (6.16) for Q and for W yields the relation

$$\Delta^* Q = ik(W - \overline{W}) \qquad (6.17)$$

where $\overline{W}$ is the constant term in the Legendre expansion (6. 16). The same relation (with the same $k=\omega/c$) is obtained for the value of these functions in the earth.

7. *Approximate Expressions for the Field*

The series for our functions P and Q are similar to the series for U which was summed approximately in Section 2. In addition, P is connected to U by the relation (6.09). Therefore, it is not necessary to repeat anew the arguments which have led us to the summation of the series for U, and we can use the results already obtained. To determine P we make use of the relation

$$\Delta^* P = \frac{\partial U}{\partial b} + \frac{U}{b} - \left(\frac{\partial U}{\partial b} + \frac{U}{b}\right)^0 \qquad (7.01)$$

and of the value (2.23) for U:

$$U = \frac{e^{iks}}{\sqrt{[sa\sin(s/a)]}} \cdot V(x, y_1, y_2, q) \qquad (7.02)$$

It is easy to see that in the approximation in which formula (7.02) is valid, the application of the operator Δ^* to functions of the type U or P is equivalent to multiplication by$(-k^2a^2)$. On the other hand, on the right side of (7.01) the constant term is to be neglected. We can also neglect the term U/b in comparison with the derivative $\partial U/\partial b$ and express this derivative according to (2.15) as the derivative with respect to y_2. Equation (7.01) then gives

$$-k^2a^2P = \frac{k}{m}\frac{\partial U}{\partial y_2} \qquad (7.03)$$

or

$$P = -\frac{1}{ka^2m}\frac{\partial U}{\partial y_2} \qquad (7.04)$$

Similarly, using (6.17) one can express Q in terms of W. We obtain

$$Q = -\frac{i}{ka^2} W. \tag{7.05}$$

We have already derived an approximate expression for U, namely (7.02). A similar expression can be derived for W. The series (6.16) for W differs from the series (1.07) for U only in that the coefficient A_n, defined by (1.08), is replaced by the coefficient B_n, defined by (6.13). With the same degree of accuracy to which formula (2.13) is valid, we can write

$$B_n = i\,\frac{v'(t)-q_1 v(t)}{w_1'(t)-q_1 w_1(t)} \tag{7.06}$$

where

$$q_1 = q\eta = im(\eta-1)^{1/2} \tag{7.07}$$

Thus the whole function W differs from U only in that q is replaced by q_1. We have

$$W = \frac{e^{iks}}{\sqrt{[sa\sin(s/a)]}}\,V(x, y_1, y_2, q_1) \tag{7.08}$$

In practice, one can put $q_1 = \infty$ in all cases. Then the series for V takes the form (3.11).

We must now insert the expressions obtained into the formulae for the field. To do this, we first find the electric and magnetic Hertz functions u and v. From (6.01) we have

$$u = \frac{i}{am}\,\frac{\partial U}{\partial y_2}\cos\varphi \tag{7.09}$$

$$v = -\frac{1}{a}\,W\sin\varphi \tag{7.10}$$

When substituting these expressions into the formulae (5.12) and (5.13), for the field we retain only the principal terms and we neglect quantities of order $1/m^2$ in comparison with unity. We then obtain the following simple expressions:

$$\begin{aligned} E_r &= -k^2 au = -\frac{ik^2}{m}\,\frac{\partial U}{\partial y_2}\cos\varphi \\ E_\vartheta &= 0 \\ E_\varphi &= k^2 av = -k^2 W\sin\varphi \end{aligned} \tag{7.11}$$

$$\begin{aligned} H_r &= k^2 av = -k^2 W\sin\varphi \\ H_\vartheta &= \frac{ik^2 a}{m}\,\frac{\partial v}{\partial y_1} = -\frac{ik^2}{m}\,\frac{\partial W}{\partial y_1}\sin\varphi \\ H_\varphi &= k^2 au = \frac{ik^2}{m}\,\frac{\partial U}{\partial y_2}\cos\varphi \end{aligned} \tag{7.12}$$

These expressions give the field in a set of units where the moment of the electric dipole is taken to be unity. To obtain the field in conventional units, these expressions must be multiplied by the value of the electric moment.

We now compare these relations with those for a vertical magnetic dipole with unit moment. According to (6.15) we must put $u = 0$; $bv = W$. We obtain

$$E_r = 0; \qquad E_\vartheta = 0; \qquad E_\varphi = k^2 W \tag{7.13}$$

$$H_r = k^2 W; \qquad H_\vartheta = \frac{ik^2}{m}\frac{\partial W}{\partial y_1}; \qquad H_\varphi = 0 \tag{7.14}$$

Thus in the plane perpendicular to the electric dipole, its field either coincides with the field of a vertical magnetic dipole ($\varphi = 3\pi/2$), or differs from it in sign ($\varphi = \pi/2$).

In conclusion we may make a remark about the character of the field at different distances from the source. At finite values of the reduced horizontal distance x, the functions U and W are of the same order. Since the formulae (7.11) and (7.12) contain the large parameter m, then at such distances the different field components will be of different orders. The electric field will be nearly horizontal and the magnetic field nearly vertical (the ratio of "small" components to "large" will be of the order of $1/m$). However, in the region of deep shadow W will decrease more rapidly than U. Indeed, the decrease of these functions is characterized by the factors

$$e^{ixt_1^0} \text{ (for } W) \qquad \text{and} \qquad e^{ixt_1'} \text{ (for } U)$$

where t_1^0 and t_1' are those roots of the equations

$$w_1'(t_1^0) - q_1 w_1(t_1^0) = 0; \qquad w_1'(t_1') - q w_1(t_1') = 0$$

which have the smallest moduli.

For soil with good conductivity we may set $q = 0$, $q_1 = \infty$ and then

$$t_1^0 = 2{\cdot}338\, e^{i\frac{\pi}{3}}; \qquad t_1' = 1{\cdot}019\, e^{i\frac{\pi}{3}}$$

so that the imaginary part of t_1^0 will be larger than the imaginary part of t_1'. The same relation holds also in the general case, because we have always $|q_1| \gg |q|$. Therefore the attenuation of W will increase more rapidly than the attenuation of U, and for sufficiently large x, the terms with U can, in spite of the small factor $1/m$, become predominant over the terms with W. This means for the electric field a constant transition from horizontal to vertical polarization (and a reverse transition for the magnetic field).

CHAPTER 13

PROPAGATION OF THE DIRECT WAVE AROUND THE EARTH WITH DUE ACCOUNT FOR DIFFRACTION AND REFRACTION†

INTRODUCTION

For a homogeneous surface of the earth the propagation of radiowaves depends mainly on the following three circumstances: diffraction around the convex surface of the earth, refraction in the lower layers of the atmosphere, and reflection from the ionosphere. At short distances, of the order of a hundred or several hundreds of kilometers, reflection from the ionosphere is of no importance. But at distances of the order of a thousand or several thousands of kilometers the reflection from the ionosphere begins to play a substantial part, because of the superposition on the direct wave of the reflected waves which may have a substantially greater intensity than the direct wave.

However, even at these great distances it is possible, under certain conditions, to separate out the direct wave and to observe it independently. Its study is of great practical interest for interference methods of determining distances. For this reason the development of a theory which would give the amplitude and phase of the direct wave up to very large distances, presents a problem important for practical purposes.

The theory of the direct wave must take account of both diffraction and refraction. Owing to the complexity of the problem, in the majority of theoretical investigations atmospheric refraction was, however, either not taken into account at all or was treated very crudely, by methods of geometrical optics. The concept of the equivalent radius of the earth, though very important in this problem, has not received adequate theoretical foundation. The concept has been introduced on the basis of considerations of bent rays, and yet, in the region of the penumbra and particularly

† Fock, 1948.

in the region of the shadow, the concept of a ray loses its significance altogether. In connection with this difficulty the conditions under which the replacement of the earth's radius by an equivalent radius is permissible, have not been made clear.

In this paper we shall give an approximate solution of the Maxwell equations for the Hertz vector which will take account of both the diffraction and the refraction. This solution is valid for very general assumptions regarding the variations of the refractive index of the air with height.

In certain important cases this solution can be expressed in terms of functions introduced in our solution of the problem of propagation of radio waves in a homogeneous atmosphere. These functions are already partially tabulated; in cases for which tables are available the computation of the field taking account of refraction does not require much work. Incidentally, the premises for the use of the concept of the equivalent radius of the earth will be elucidated and we shall show that this concept is applicable also in the region of the shadow and the penumbra (where geometrical optics is not applicable).

The present paper is a direct continuation of our previous work on diffraction of radio waves around the earth's surface.

1. *Differential Equations and Boundary Conditions of the Problem*

Let us denote by r, ϑ, and φ, spherical coordinates with the origin at the centre of the earth's globe and with the polar axis passing through the radiating dipole. We shall assume the dipole to be situated on the surface of the earth and we shall investigate the field in the air. The radius of the earth we shall denote by a. The dielectric constant of the air will be assumed to be a function of the height $h = r-a$ above the surface of the earth

$$\varepsilon = \varepsilon(h), \quad h = r-a \tag{1.01}$$

As in the case of the homogeneous atmosphere, the field components in the air may be expressed by the Hertz function U. We have

$$E_r = \frac{1}{r \sin\vartheta} \cdot \frac{\partial}{\partial\vartheta}\left(\sin\vartheta \frac{\partial U}{\partial\vartheta}\right) \tag{1.02}$$

$$E_\vartheta = -\frac{1}{\varepsilon \cdot r} \cdot \frac{\partial}{\partial r}\left(\varepsilon r \frac{\partial U}{\partial\vartheta}\right) \tag{1.03}$$

$$H_\varphi = -ik_0\varepsilon \frac{\partial U}{\partial\vartheta} \tag{1.04}$$

and the remaining components are equal to zero. The time dependence of the field will be assumed of the form $e^{-i\omega t}$ where

$$\frac{\omega}{c} = k_0 = \frac{2\pi}{\lambda_0} \tag{1.05}$$

Here λ_0 is the wave length in vacuum (in our problem it is necessary to distinguish it from that in the air). The value of the dielectric constant of the air at the surface of the earth will be denoted by $\varepsilon_0 = \varepsilon(0)$ and we shall denote by

$$k = \frac{2\pi}{\lambda} = k_0\sqrt{\varepsilon_0} \tag{1.06}$$

the value of the wave number at the surface of the earth.

The field, expressed by the formulae (1.02) to (1.04) will satisfy Maxwell's equations if the function U satisfies the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\cdot\frac{\partial}{\partial r}(\varepsilon r U)\right)+\frac{1}{r\sin\vartheta}\cdot\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial U}{\partial\vartheta}\right)+k_0^2\varepsilon r U = 0 \tag{1.07}$$

Let us introduce a new function

$$U_1 = \varepsilon\cdot r\sqrt{(\sin\vartheta)}\cdot U \tag{1.08}$$

This function must satisfy the equation

$$\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\cdot\frac{\partial U_1}{\partial r}\right)+\frac{1}{\varepsilon\cdot r^2}\left[\frac{\partial^2 U_1}{\partial\vartheta^2}+\left(\frac{1}{4}+\frac{1}{4\sin^2\vartheta}\right)U_1\right]+k_0^2 U_1 = 0 \tag{1.09}$$

The field at the surface of the earth must satisfy Leontovich's condition

$$E_\vartheta = -\frac{1}{\sqrt{\eta}}H_\varphi \tag{1.10}$$

where

$$\eta = \varepsilon_2 + i\frac{4\pi\sigma_2}{\omega} \tag{1.11}$$

is the complex dielectric constant of the soil. Leontovich's condition will be satisfied if the function U satisfies the condition

$$\frac{\partial U_1}{\partial r} = -\frac{ik_0\varepsilon_0}{\sqrt{\eta}}U_1 \qquad (\text{for } r = a) \tag{1.12}$$

In the function U_1 let us separate out a rapidly varying factor by putting

$$U_1 = e^{ika\vartheta}U_2 = e^{iks}U_2 \tag{1.13}$$

where k is the quantity (1.06) and $s = a\vartheta$ is the length of the arc of the earth's globe from the point where the dipole is situated to the point over which the field is being computed. For the function U_2 we obtain the equation

$$\frac{\partial^2 U_2}{\partial r^2}+2i\frac{k}{a}\cdot\frac{\partial U_2}{\partial\vartheta}+k^2\left(\frac{\varepsilon}{\varepsilon_0}-\frac{a^2}{r^2}\right)U_2$$
$$=\frac{\varepsilon'}{\varepsilon}\cdot\frac{\partial U_2}{\partial r}-\frac{1}{r^2}\left[\frac{\partial^2 U_2}{\partial\vartheta^2}+\left(\frac{1}{4}+\frac{1}{4\sin^2\vartheta}\right)U_2\right] \tag{1.14}$$

where ε' denotes the derivative of $\varepsilon(h)=\varepsilon(r-a)$ with respect to r.

The equation (1.14) is so written that the left-hand side contains the main terms while the right-hand side contains correction terms, which, as we shall show, may be replaced by zero.

When estimating the order of magnitude of the derivatives we may use the results obtained for the case of a homogeneous atmosphere. If we introduce the large parameter

$$m=\sqrt[3]{\left(\frac{ka}{2}\right)} \tag{1.15}$$

then we will have

$$\frac{\partial U_2}{\partial r}=0\left(\frac{k}{m}U_2\right);\qquad \frac{\partial U_2}{\partial\vartheta}=0\left(\frac{ka}{m^2}U_2\right) \tag{1.16}$$

where the symbol 0 stands for "of the order of".

On the other hand, if we exclude from our considerations the ionosphere (where ε may become zero) then the gradient of the logarithm of ε will be of the order of the curvature of the earth's surface so that

$$\frac{\varepsilon'}{\varepsilon}=0\left(\frac{1}{a}\right) \tag{1.17}$$

From this it will be seen that separate terms on the left-hand side of (1.14) will be not less than order $(k^2/m^2)\cdot U_2$ while on the right-hand side the terms containing the derivatives will be of the order $(k^2/m^4)\cdot U_2$. The term containing $\sin^2\vartheta$ in the denominator will be also small if

$$ks \gg m \tag{1.18}$$

Neglecting quantities of the order $1/m^2$ as compared with unity, we can substitute zero for the right-hand side of equation (1.14) after which

we obtain

$$\frac{\partial^2 U_2}{\partial r^2}+2i\frac{k}{a}\cdot\frac{\partial U_2}{\partial \vartheta}+k^2\left(\frac{\varepsilon}{\varepsilon_0}-\frac{a^2}{r^2}\right)U_2 = 0 \tag{1.19}$$

This is the parabolic equation of our problem which resembles in form the Schrödinger equation of quantum mechanics. In this equation, we can make further simplification by using the approximate equality

$$1-\frac{a^2}{r^2} = 2\frac{h}{a} \tag{1.20}$$

Introducing, in place of the angle ϑ the length of the arc $s=a\vartheta$ and taking s and h for independent variables, we obtain

$$\frac{\partial^2 U_2}{\partial h^2}+2ik\frac{\partial U_2}{\partial s}+k^2\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}+\frac{2h}{a}\right)U_2 = 0 \tag{1.21}$$

The boundary condition for U_2 at the surface of the earth will be the same as for U_1, namely

$$\frac{\partial U_2}{\partial h} = -ik\sqrt{\left(\frac{\varepsilon_0}{\eta}\right)}U_2 \qquad (\text{for } h = 0) \tag{1.22}$$

The condition at infinity ($h\to\infty$) may be obtained from the consideration of the phase of the Hertz function. If we put

$$U = |U|\,e^{i\Phi} \qquad \text{and} \qquad U_2 = |U_2|\,e^{i(\Phi-ks)} \tag{1.23}$$

then, since we are considering the wave coming from the source, the phase Φ must increase with increasing height h. From this we obtain the condition

$$\frac{\partial \Phi}{\partial h} > 0 \tag{1.24}$$

which must be fulfilled, at least, for sufficiently large values of h.

Moreover, the Hertz function U, and also the function U_2 must remain finite and continuous throughout the whole space with the exception of the region adjoining the source.

To determine the solution of the equation (1.21) uniquely it remains to formulate the condition that must be satisfied by the function U_2 in the region adjacent to the source. First, it is obvious that in the immediate vicinity of the source equation (1.18) does not hold, and equation (1.21) itself is no longer correct. Therefore the condition at the source must be

replaced by an equivalent condition in the wave zone. For example, we may take a region where the reflection formula applies, and obtain the desired condition by requiring that the solution we seek should go over in this region into the reflection formula.

The reflection formula has the form

$$U = \frac{e^{ikR}}{R}(1+f) \tag{1.25}$$

where f is the Fresnel coefficient. Since we are using the Leontovich boundary conditions (1.10) we assume that $|\eta| \gg 1$. If, in addition, we take $h \ll s$, i.e. if we consider small angles of inclination of the ray, then we can put

$$R = s+\frac{h^2}{2s}; \qquad f = \frac{h\sqrt{\eta}-s}{h\sqrt{\eta}+s} \tag{1.26}$$

Substituting these expressions in (1.25), we come to the conclusion that in the region where the reflection formula applies, the function

$$U_2 = e^{-iks}\, \varepsilon r \sqrt{(\sin\vartheta)}\, U \tag{1.27}$$

must reduce to

$$U_2 = \frac{\varepsilon_0\sqrt{a}}{\sqrt{s}} \cdot \frac{2h\sqrt{\eta}}{h\sqrt{\eta}+s}\, e^{i\frac{kh^2}{2s}} \tag{1.28}$$

Mathematically, this condition is equivalent to the requirement that for $s \to 0$ and $h > 0$ the function U_2 should have a singularity characterized by the equation

$$\lim_{s\to 0}\left(U_2 - \frac{2\varepsilon_0\sqrt{a}}{\sqrt{s}}\, e^{ik\frac{h^2}{2s}}\right) = 0 \tag{1.29}$$

A more detailed discussion of the condition (1.29) can be found in Chapter 11.

Let us remark that in place of conditions (1.29) and (1.28) we could have set up a still more stringent condition, requiring that in the near region where the curvature of the earth's surface and the inhomogeneity of the atmosphere ceases to have influence and where the formula of Weyl–van der Pol† is applicable, our solution should go over into the solution of Weyl–van der Pol.

† The range of application of the Weyl–van der Pol formula was investigated in more detail in Ref. 22 and in Chapters 10 and 11.

2. *Introduction of Dimensionless Quantities*

The differential equation for the function U_2, derived above, is of the form

$$\frac{\partial^2 U_2}{\partial h^2}+2ik\frac{\partial U_2}{\partial s}+k^2\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}+\frac{2h}{a}\right)U_2=0 \tag{2.01}$$

Let us consider the coefficient of U_2 in this equation. Denoting by ε_0' the value of the gradient of the dielectric constant near the surface of the earth, we can separate out of the expression for ε the linear term and write the coefficient of U_2 in the form

$$k^2\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}+\frac{2h}{a}\right)=k^2\left[\frac{\varepsilon-\varepsilon_0-\varepsilon_0'h}{\varepsilon_0}+\left(\frac{2}{a}+\frac{\varepsilon_0'}{\varepsilon_0}\right)h\right] \tag{2.02}$$

Now we put

$$\frac{1}{a^*}=\frac{1}{a}+\frac{\varepsilon_0'}{2\varepsilon_0} \tag{2.03}$$

The quantity (2.03) is the difference between the curvature of the earth's surface and the curvature of the ray, and the inverse quantity a^* is commonly designated as the equivalent radius of the earth. With the notation (2.03) we can write the formula (2.02) in the form

$$k^2\left(\frac{\varepsilon-\varepsilon_0}{\varepsilon_0}+\frac{2h}{a}\right)=\frac{2k^2}{a^*}h(1+g) \tag{2.04}$$

where

$$g=\frac{a^*}{2\varepsilon_0}\left(\frac{\varepsilon-\varepsilon_0}{h}-\varepsilon_0'\right) \tag{2.05}$$

As can be seen from (2.05), the quantity g is the difference between the height average of the gradient of the dielectric constant in air (calculated in the interval from the surface of the earth to the given height) and the value of this gradient at the earth's surface, this difference being expressed in dimensionless units. In a normal atmosphere the quantity g is positive but in case of a temperature inversion it may become negative and then only from a certain height upwards will it become positive again. The absolute value of the quantity g is usually not greater than 0·2 or 0·3. For $h\to\infty$ the theoretical value of g is $(a^*-a)/a$ and for $h=0$ we have $g=0$. In the case of the normal atmosphere the quantity g changes very slowly, but in case of an inversion its change is more rapid.

Inserting expression (2.04) into the differential equation (2.01) we obtain

$$\frac{\partial^2 U_2}{\partial h^2}+2ik\frac{\partial U_2}{\partial s}+\frac{2k^2}{a^*}h(1+g)U_2 = 0 \tag{2.06}$$

For the investigation of equation (2.06) it is convenient to change from h and s to dimensionless quantities. For this purpose we shall introduce vertical and horizontal scales

$$h_1 = \sqrt[3]{\left(\frac{a^*}{2k^2}\right)}; \qquad s_1 = \sqrt[3]{\left(\frac{2a^{*2}}{k}\right)} \tag{2.07}$$

and put

$$\frac{h}{h_1} = y; \qquad \frac{s}{s_1} = x \tag{2.08}$$

In order to simplify the condition (1.29), we will introduce a new dimensionless function W_1, putting

$$U_2 = \frac{\varepsilon_0\sqrt{a}}{\sqrt{s_1}}W_1 \tag{2.09}$$

We also put

$$q = ikh_1\sqrt{\left(\frac{\varepsilon_0}{\eta}\right)} = i\sqrt[3]{\frac{(ka^*)}{2}}\sqrt{\left(\frac{\varepsilon_0}{\eta}\right)} \tag{2.10}$$

With these new notations, the differential equation, the boundary condition, and the condition determining the singularity assume the form

$$\frac{\partial^2 W}{\partial y^2}+i\frac{\partial W_1}{\partial x}+y(1+g)W_1 = 0 \tag{2.11}$$

$$\frac{\partial W_1}{\partial y}+qW_1 = 0 \qquad (\text{for } y = 0) \tag{2.12}$$

$$\lim_{x\to 0}\left(W_1-\frac{2}{\sqrt{x}}e^{i\frac{y^2}{4x}}\right) = 0 \qquad (y > 0) \tag{2.13}$$

Moreover, the condition for the phase $\Phi = ks+\text{arc } W_1$, namely

$$\frac{\partial \Phi}{\partial y} > 0 \qquad (\text{for } y \gg 1) \tag{2.14}$$

remains.

The quantity g entering in equation (2.11) was defined above (formula (2.05)) as a function of the height h. We introduce a certain height

h_0 characterizing the rate of change of the gradient of the dielectric constant of air, for example, the height interval within which the gradient changes by a factor $e=2{\cdot}718$. (For the normal atmosphere $h_0=7400$ m; in other cases it is possible to indicate only the order of magnitude of h_0, which is all that we need). The quantity g may be regarded as a function of the ratio h/h_0. We write

$$g = g\left(\frac{h}{h_0}\right); \qquad g(0) = 0 \tag{2.15}$$

and assume that the derivative of this function with respect to its argument will be of the order of unity. When the dimensionless quantities (2.08), are introduced, the quantity g must be regarded as a function of y. Since $h=h_1y$ we shall have

$$g = g\,(\beta y) \tag{2.16}$$

where

$$\beta = \frac{h_1}{h_0} = \frac{1}{h_0}\sqrt[3]{\left(\frac{a^*}{2k^2}\right)} \tag{2.17}$$

In the following we shall regard the parameter β as a small quantity. To estimate its order of magnitude we put $h_0=7400$ m (normal atmosphere) and replace the equivalent radius a^* by the geometrical radius a. Then for $\lambda=1$ m, $\lambda=10$ m, $\lambda=100$ m, and $\lambda=1000$ m the corresponding values of β are $\beta=0{\cdot}006$, $\beta=0{\cdot}027$, $\beta=0{\cdot}13$, $\beta=0{\cdot}58$. In case of an inversion, the quantity h_0 will be much less and the parameter β will be small only for shorter wavelengths.

3. *Solution of the Equation*

If in the equation

$$\frac{\partial^2 W_1}{\partial y^2} + i\frac{\partial W_1}{\partial x} + y[1+g(\beta y)]W_1 = 0 \tag{3.01}$$

we suppose that $\beta=0$, then, since $g(0)=0$, the function g will become identically equal to zero, and the equation will reduce to that which was discussed and solved (with the boundary conditions (2.12) and (2.13)) in previous chapters where the case of a homogeneous atmosphere was considered. It is important to notice, however, that the condition $\beta=0$ corresponds not to the assumption of the homogeneity of the atmosphere, but to the more general assumption of constancy of the gradient of the dielectric constant. In this more general case the formulae obtained are

the same as in the case of a homogeneous atmosphere with the only difference that in the expressions for x, y, and q the radius of the earth a is replaced by the equivalent radius a^*. Thus the smallness of the quantity β determines the degree of approximation with which it is possible to apply (for finite values of y) the concept of the equivalent radius.

We have obtained the solution for $\beta=0$ in the form of an integral containing the complex Airy function. The latter is that solution of the differential equation

$$w''(t) = tw(t) \tag{3.02}$$

which has, for large negative values of t, the asymptotic expression

$$w(t) = e^{i\frac{\pi}{4}} \cdot (-t)^{-\frac{1}{4}} e^{i\frac{2}{3}(-t)^{3/2}} \tag{3.03}$$

The solution for $g(\beta y)=0$ has the form

$$W_1 = e^{-i\frac{\pi}{4}} \frac{1}{\sqrt{\pi}} \int_C e^{ixt} \cdot \frac{w(t-y)}{w'(t)-qw(t)} \, dt \tag{3.04}$$

where the contour C runs from $t=i\infty$ to $t=0$ and from $t=0$ to $t=\infty \cdot e^{i\alpha}$ $(0<\alpha<\pi/3)$ enclosing all the roots of the denominator in the integrand. (This contour can, of course, be replaced by some other equivalent contour.) This solution coincides with that which was obtained in Chapter 10 and in Ref. 22.

Using a similar method we shall try to find a solution of our equations for the general case of $\beta \neq 0$. We shall not suppose from the beginning that β is small; this assumption will be introduced only later, in order to simplify the general solution obtained.

The equation (3.01) admits separation of the variables. Particular solutions of equation (3.10) having the form of a function of x multiplied by a function of y and containing an arbitrary parameter t, can be written as

$$W_1 = e^{ixt} f(y, t) \tag{3.05}$$

where $f(y, t)$ satisfies the equation

$$\frac{d^2 f}{dy^2} + [y - t + yg(\beta y)] f = 0 \tag{3.06}$$

From the theory of differential equations it is known that if the initial values (i.e. its values for $y=0$) of the function f and its derivative with respect to y are integral functions of the parameter t, then the solution of equation (3.06) will be an integral transcendental function of t. We shall

mean by $f(y, t)$ the solution of the equation (3.06) which is an integral transcendental function of t and which has, for large values of the difference $y-t$ (or of its real part), the asymptotic representation

$$f(y, t) = \frac{C\mathrm{e}^{i\frac{\pi}{4}}}{\sqrt[4]{[y-t+yg(\beta y)]}} \exp\left\{i\int_{\tau}^{y} \sqrt{[u-t+ug(\beta u)]}\, \mathrm{d}u\right\} \tag{3.07}$$

The lower limit τ in the integral which appears in the exponential may be taken arbitrarily. The coefficient C may be a function of the parameter t. The phase factor $\mathrm{e}^{i\frac{\pi}{4}}$ is added in order that, for $g=0$ and $\tau=\mathrm{t}$, the expression (3.07) would go over into the asymptotic expression for the function

$$f(y, t) = Cw(t-y) \tag{3.08}$$

The expression (3.07) was taken in accordance with the requirement $\partial\Phi/\partial y > 0$ imposed on the phase.

Calling $f(y, t)$ the solution of equation (3.06) just defined, we shall consider the expression

$$W_1 = \mathrm{e}^{i\frac{3\pi}{4}} \frac{1}{\sqrt{\pi}} \int_{\Gamma} \mathrm{e}^{ixt} \frac{f(y, t)}{\left(\dfrac{\partial f}{\partial y}+qf\right)_{y=0}} \mathrm{d}t \tag{3.09}$$

where the contour Γ is similar to the contour C in the integral (3.04).

In the first place we notice that the conditions laid down above determine the integrand in (3.09) uniquely, because the factor C which remained undetermined in (3.07) cancels out.

Further, the integrand in (3.09) is a meromorphic function of the complex variable t; the only singular points in it are the roots of the denominator.

It would be difficult to carry out the investigation of the roots of the denominator in (3.09) with full rigour. For such investigation it would be necessary to know the behaviour of the function $g(\beta y)$ for complex values of y near the straight half-line arc $y=\pi/3$. However, on the basis of certain not fully rigorous considerations which we shall omit here, it can be expected that if in the complex region indicated the function $g(\beta y)$ will remain small $\left(\text{for example } |g| < \frac{1}{2}\right)$, then the roots will be distributed in the same way as in the case $g=0$, i.e., in the first quadrant of the t-plane near the straight half-line arc $t=\pi/3$. In any case it will be true for small values of the parameter β.

We need also the behaviour of the function $f(y, t)$ for positive values of $t-y$ (and also in a certain sector of the t-plane including the positive real axis). The desired asymptotic expression will be obtained by analytical continuation of expression (3.07) through the third and fourth quadrants of the t-plane, because the roots of $f(y, t)$ are situated in the first quadrant. It will have the form

$$f(y, t) = \frac{C}{\sqrt[4]{[t-y+yg(\beta y)]}} \cdot \exp\left\{ \int_y^\tau \sqrt{[t-u+ug(\beta u)]}\, du \right\} \qquad (3.10)$$

If we put here $g=0$ and take $\tau=t$, then this expression will lead, as did (3.07), to the asymptotic expression for the function (3.08).

Knowing the distribution of the poles and the behaviour of the integrand on both sides of the region where the poles are situated it is possible to draw the contour in the integral (3.09) in such a way that it encloses all the roots of the denominator and has two branches going to infinity. For the initial branch of the contour (coming from infinity) the asymptotic relation (3.07) will hold and for the terminal branch (going to infinity) the expression (3.10) applies. The integral taken along this contour will be convergent.

The preceding discussion has shown that the expression (3.09) for the function W_1 is well-defined.

We now show that it satisfies all the required conditions. First, it is clear that it satisfies the differential equation (3.01) because the latter is satisfied by the integrand. Further, it satisfies the boundary condition (2.12)

$$\frac{\partial W_1}{\partial y} + qW_1 = 0 \qquad (\text{for } y = 0) \qquad (3.11)$$

Performing the differentiation in (3.09) under the integral sign and then putting $y = 0$, we see that the numerator of the fraction will cancel with the denominator and the new integrand will be holomorphic, so that the integral will vanish. Next, the integral is convergent and, therefore, finite for all positive values of x and y. Finally it is not difficult to verify that it will satisfy the condition for the phase ($\partial\Phi/\partial y > 0$)

It remains to verify whether the expression (3.09) has the singularity near $x=0$, required by the condition (2.13), or, what is equivalent, to verify whether at short distances from the source it gives the Weyl–van der Pol formula or the reflection formula.

With the aid of the asymptotic expression (3.07) and (3.10) for $f(y, t)$, it is possible to show that if x and y are small, and the quotient y/x is

large, then the principal section of the integration path will lie in the region of large negative values of t. (The original contour Γ can be deformed so as to pass through this region). Making use of the expression (3.07), we obtain for large negative values of t

$$\frac{f(y, t)}{f(0, t)} = \sqrt[4]{\left(\frac{-t}{y-t+yg(\beta y)}\right)} \cdot \exp\left[i\int_0^y \sqrt{(u-t+ug)}\,\mathrm{d}u\right] \tag{3.12}$$

From this we have

$$\frac{1}{f} \cdot \frac{\partial f}{\partial y} = i\sqrt{[y-t+yg(\beta y)]} \tag{3.13}$$

and

$$\left(\frac{1}{f} \cdot \frac{\partial f}{\partial y} + q\right)_{y=0} = i\sqrt{-t}+q \tag{3.14}$$

But when y is small the terms $yg(\beta y)$ are small compared with y and we can write in place of (3.12)

$$\frac{f(y, t)}{f(0, t)} = \sqrt[4]{\left(\frac{-t}{y-t}\right)} \exp\left[i\int_0^y \sqrt{(u-t)}\,\mathrm{d}u\right] \tag{3.15}$$

Now, the same asymptotic expressions will be obtained for the region considered if in place of $f(y, t)$ we take

$$f(y, t) = w(t-y) \tag{3.16}$$

But after such a substitution the integral (3.09) becomes equal to (3.04) and the latter gives, for small values of x, y the Weyl–van der Pol formula and the reflection formula; it satisfies also the condition (2.13).

We can also verify this in a more direct way. Introducing the variable of integration $p = \sqrt{(-t)}$ and neglecting the quantities y and y^2 as compared with p we find

$$\frac{f(y, -p^2)}{f(0, -p^2)} = \mathrm{e}^{iyp} \tag{3.17}$$

and

$$\left(\frac{1}{f} \cdot \frac{\partial f}{\partial y} + q\right)_{y=0} = ip+q \tag{3.18}$$

The substitution of these quantities into the integral (3.09) gives

$$W_1 = \mathrm{e}^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \int_{\Gamma_2} \mathrm{e}^{-ixp^2+iyp} \cdot \frac{p\,\mathrm{d}p}{p-iq} \tag{3.19}$$

where the contour Γ_2 intersects the positive real axis in the p-plane from below upwards (in the vicinity of the point $p=y/2x$). If we compute the integral (3.19) without neglecting anything, we arrive at the Weyl–van der Pol formula. If we compute it by the method of stationary phase we arrive at the reflection formula. If we neglect the quantity $|q|$ in comparison with $y/2x$ we obtain an expression which reduces to zero the left-hand side of (2.13) even before going over to the limit.

We have thus proved that the expression (3.19) for W_1 represents the desired solution of our problem.

4. *Investigation of the Solution for the Region of Direct Visibility*

Instead of the function W_1, it is sometimes more convenient to consider another function that differs from W_1 in (3.09) by the factor $\sqrt{x}$. We put

$$V(x, y, q) = \mathrm{e}^{i\frac{3\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_{\Gamma} \mathrm{e}^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y}+qf\right)_0} \mathrm{d}t \tag{4.01}$$

Remembering the connection between the functions U, U_1, U_2, and W_1, given by the formulae (1.08), (1.13), and (2.09), and neglecting the distinction between r and a and between ε and ε_0 when these quantities enter as factors of U we can write

$$U = \frac{\mathrm{e}^{iks}}{\sqrt{\left\{as \sin \frac{s}{a}\right\}}} V(x, y, q) \tag{4.02}$$

where s is, as before, the horizontal distance, measured along the arc of the earth's surface, and x, y, and q, are connected with s, h, η by the relations

$$x = \frac{s}{s_1}; \qquad y = \frac{h}{h_1}; \qquad q = i \sqrt[3]{\left(\frac{ka^*}{2}\right)} \sqrt{\left(\frac{\varepsilon_0}{\eta}\right)} \tag{4.03}$$

and

$$s_1 = \sqrt[3]{\left(\frac{2a^{*2}}{k}\right)}; \qquad h_1 = \sqrt[3]{\left(\frac{a^*}{2k^2}\right)} \tag{4.04}$$

If s is small as compared with the radius of the earth, then it is permissible to write instead of $\sin s/a$ simply s/a. However, since the formulae remain valid up to very great distances where the difference between the sine and the arc becomes significant, we retain the factor $\sin s/a$ under the radical in (4.02).

The function $V(x, y, q)$ may be called the attenuation factor; in cases when we can put $g=0$ and use the concept of the equivalent radius, equation (4.01) for V takes the form

$$V(x, y, q) = e^{-i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_{\Gamma} e^{ixt} \frac{w(t-y)}{w'(t)-qw(t)} \, dt \tag{4.05}$$

The function (4.05) was investigated in detail in Ref. 22. (Section 6) and in Chapter 10 and partially tabulated (for $q=0$). The investigation which follows is in many respects similar to the investigation in Chapter 10.

In the present section we shall consider the illuminated (line-of-sight) region. In the line-of-sight region not too near to the horizon geometrical optics becomes applicable. If we make use of the expression (3.12) and introduce the variable of integration $p=\sqrt{(-t)}$ we obtain for V an integral of the form

$$V = e^{-i\frac{3\pi}{4}} \frac{2}{\sqrt{\pi}} \sqrt{x} \int e^{i\omega} \sqrt[4]{\frac{p^2}{y+p^2+yg(\beta y)}} \cdot \frac{p \, dp}{p-iq} \tag{4.06}$$

where we have put for brevity†

$$\omega = -xp^2 + \int_0^y \sqrt{[u+p^2+ug(\beta u)]} \, du \tag{4.07}$$

Calculating the integral by the method of stationary phase, we find the extremum of the phase from the equation $\partial\omega/\partial p = 0$ and obtain

$$x = \frac{1}{2} \int_0^y \frac{du}{\sqrt{[u+p^2+ug(\beta u)]}} \tag{4.08}$$

After some calculations we arrive at the expression

$$V = e^{i\omega} \frac{2p}{p-iq} \sqrt{\left(2x \frac{\partial p}{\partial y}\right)} \tag{4.09}$$

In this formula p means a function of x and y defined by equation (4.08). For $g=0$ and also for small values of x and y, we have

$$p = \frac{y-x^2}{2x} \tag{4.10}$$

† This phase ω is not to be confused with the angular frequency ω.

and the expression under the sign of the radical in (4.09) becomes equal to unity. Formula (4.09) is valid for large and positive values of p.

Our formulae permit a simple interpretation from the point of view of geometrical optics. Indeed, the complete phase

$$\Phi = ks+\omega \tag{4.11}$$

of the function U is a solution of the eikonal equation

$$\left(1+\frac{h}{a}\right)^2\left(\frac{\partial\Phi}{\partial h}\right)^2+\left(\frac{\partial\Phi}{\partial s}\right)^2 = k^2\left(1+\frac{h}{a}\right)^2\frac{\varepsilon}{\varepsilon_0} \tag{4.12}$$

which, after neglecting small quantities, leads to the following equation for ω:

$$\left(\frac{\partial\omega}{\partial h}\right)^2+2k\cdot\frac{\partial\omega}{\partial s} = \frac{2k^2}{a^*}h(1+g) \tag{4.13}$$

The right-hand side of this equation is the quantity (2.04). Introducing the variables x and y we obtain from (4.13)

$$\left(\frac{\partial\omega}{\partial y}\right)^2+\frac{\partial\omega}{\partial x} = y+yg(\beta y) \tag{4.14}$$

Formula (4.08) is the equation of the trajectory of the ray passing through the origin of coordinates, and the quantity p is the parameter of this trajectory. The geometrical significance of the parameter p is

$$p = \sqrt[3]{\left(\frac{ka^*}{2}\right)}\cdot\cos\gamma \tag{4.15}$$

where γ is the angle between the ray in the vicinity of the source and the vertical. The complete phase Φ is the optical length of the path of the ray, reckoned from the source to the point x, y. The quantity $2p/(p-iq)$ is equal to

$$\frac{2p}{p-iq} = 1+f \tag{4.16}$$

where f is the Fresnel coefficient.

Thus, in those cases when geometrical optics is applicable, our formulae go over into the formulae of geometrical optics.

If x and y are finite formula, (4.09) is applicable in the case when the parameter p is positive and large. If x and y are small, the following addi-

tional condition becomes necessary

$$\frac{y^2}{4x} = \frac{kh^2}{2s} \gg 1 \cdot . \tag{4.17}$$

If in the case of small values of x and y and large values of p the condition (4.17) is not fulfilled, the expression (4.06) remains valid, but the integral must be calculated differently, namely ω must be replaced by $-xp^2+yp$ and the fourth root must be replaced by unity, after which the integral takes the form (3.19) (with a factor $\sqrt{x}$) and yields the Weyl-van der Pol formula.

Let us observe that if x and $\sqrt{y}$ are large, and the parameter p is small as compared with these quantities, then the equation of the trajectory (4.08) can be solved approximately for p. We have, as an approximation,

$$p = \frac{1}{2}\int_0^v \frac{\mathrm{d}u}{\sqrt{[u+ug(\beta u)]}} - x \tag{4.18}$$

Under the same conditions

$$\omega = \omega_0(y) + \frac{1}{3}p^3 \tag{4.19}$$

where

$$\omega_0(y) = \int_0^v \sqrt{[u+ug(\beta u)]}\,\mathrm{d}u \tag{4.20}$$

and the symbol p is to be understood as an abbreviation for the quantity (4.18).

The equation $p=0$ gives the geometrical boundary of the shadow. If the right-hand side of (4.18) becomes negative, then the equation (4.08) cannot be satisfied with a real p; function (4.18) (and also (4.10)) however retains its significance in this case also. This apparent discrepancy is explained by the fact that the right-hand part of (4.08) is not an analytical function of p near the point $p=0$.

We shall meet the expressions (4.18) and (4.19) in the study of the penumbra region where geometrical optics is no longer applicable.

5. *Investigation of the Solution for the Region of the Penumbra (Finite x and y)*

The region of the penumbra is characterized by the fact that within it the parameter p, defined by formula (4.10), is a positive or a negative number of the order of unity.

If x and y are finite, we may form for V a series of residues at the poles of the integrand. We shall have

$$V(x, y, q) = e^{i\frac{\pi}{4}} 2\sqrt{(\pi x)} \sum_{n=1}^{\infty} \frac{e^{ixt}}{D(t_n)} \cdot \frac{f(y, t_n)}{f(0, t_n)} \tag{5.01}$$

where we have put

$$D(t) = -\frac{1}{f(0, t)}\left(\frac{\partial^2 f}{\partial y\, \partial t} + q\frac{\partial f}{\partial t}\right)_{y=0} \tag{5.02}$$

and t is a root of the equation

$$\left(\frac{\partial f}{\partial y} + qf\right)_{y=0} = 0 \tag{5.03}$$

If β is not small then the computations with these formulae are very complicated. For this reason we shall limit ourselves in the following to the case of very small values of β. At the same time, however, we shall not suppose the product βy to be small but shall also consider large values of y (of the order $1/\beta$ and larger).

If β is small, then in computing the first roots of the function (5.03) we can replace $g(\beta y)$ by a linear function

$$g(\beta y) = g'(0)\cdot\beta y = \beta_0 y \tag{5.04}$$

The physical meaning of the coefficient β_0 is

$$\beta_0 = h_1\left(\frac{dg}{dh}\right)_0 = \frac{h_1 a^* \varepsilon_0''}{4\varepsilon_0} \tag{5.05}$$

where h_1 is the scale of height and ε_0'' is the value of the second derivative of ε with respect to height at the surface of the earth.

For small values of β_0 and finite values of y and t we can take for an approximate solution of the equation (3.06) the following function

$$f(y, t) = w(t-y) - \frac{\beta_0}{15}\left[(3y+2t)w(t-y) + (3y^2+4yt+8t^2)w'(t-y)\right] \tag{5.06}$$

Inserting this into (5.03) we find for the required root the approximate expression

$$t_n = t_n^0 + \frac{\beta_0}{15}\left[8(t_n^0)^2 - \frac{3+4t_n^0 q}{t_n^0 - q^2}\right] \tag{5.07}$$

where t_n^0 is the root of the equation

$$w'(t_n^0) - qw(t_n^0) = 0 \tag{5.08}$$

which was investigated in detail in Ref. 22 and in Chapter 10. For the function $D(t)$ we obtain from (5.06) the expression

$$D(t) = (t-q^2)\left(1-\frac{4}{3}\beta_0 t\right)+\frac{2}{3}\beta_0 q \tag{5.09}$$

The height coefficients in the formula (5.01)

$$f_n(y) = \frac{f(y, t_n)}{f(0, t_n)} \tag{5.10}$$

may be obtained by numerical integration of the differential equation

$$\frac{d^2 f_n}{dy^2}+[y-t_n+yg(\beta y)]f_n = 0 \tag{5.11}$$

with the initial conditions

$$f(0) = 1 \qquad \text{and} \qquad f'(0) = -q \tag{5.12}$$

As long as y is finite (even though x may be very large) the values of height factors obtained in this way will, for small values of β, differ but little from their values for $\beta=0$. Essential differences may arise only in the exponential factors e^{ixt}, giving the attenuation and the additional phase. For this reason it is sufficient to introduce the correction in the exponential factors only.

If no great accuracy is required, this correction may be neglected and we can take it that in the case considered the expression for $V(x, y, q)$ coincides with the one derived for the case of homogeneous atmosphere (provided that the radius of the earth is replaced by the equivalent radius). Then all the formulae and tables obtained for that case can be used.

6. *Investigation of the Solution for the Penumbra Region (Large Values of x and y)*

The case presenting the greatest interest in practice is the one where x and y are very large while the quantity

$$p = \frac{1}{2}\int_0^y \frac{\mathrm{d}u}{\sqrt{\{u+ug(\beta u)\}}}-x \tag{6.01}$$

is finite. We already pointed out that the value $p=0$ corresponds to the boundary of direct visibility (of the line-of-sight region); positive values of p correspond to the line-of-sight region and the negative values of p to the region beyond the horizon.

In this case, in computing the integral (4.01) for $V(x, y, q)$ it is necessary to keep in mind that the principal section of the integration path will correspond to finite values of t while y will be large. For this reason it is necessary to find analytical expression for $f(y, t)$ which would be valid both for very large and for finite values of $(y-t)$. This is possible if the values of β are small.

Let us introduce the quantity X, defined by the equation

$$\frac{2}{3}(-X)^{3/2} = \int_\tau^y \sqrt{[u-t+ug(\beta u)]}\, \mathrm{d}u \tag{6.02}$$

or

$$\frac{2}{3}(X)^{3/2} = \int_y^\tau \sqrt{[t-u-ug(\beta u)}\, \mathrm{d}u \tag{6.03}$$

where τ is the root of the equation

$$\tau - t + \tau g(\beta\tau) = 0 \tag{6.04}$$

For small values of β_0 and for finite values of y and t we have

$$X = t - y - \frac{\beta_0}{15}(3y^2 + 4ty + 8t^2) \tag{6.05}$$

Then the function

$$f(y, t) = \sqrt{\left(-\frac{\mathrm{d}y}{\mathrm{d}X}\right)}\, w(X) \tag{6.06}$$

will be a solution of equation (3.06) with an error of the order of β^2 for finite values of y and t and of the order of $\beta^{3/2}$ for large values of y and finite values of t. With the aid of expression (6.05) it is not difficult to verify that if the function (6.06) is expanded in powers of β_0, the terms up to β_0 inclusive in this expansion are identical with (5.06). However, the expression (6.06) holds also in those cases when (for large values of y) the expansion (5.06) is not applicable. If the quantity X is very large and negative (which is the case for large values of y) then expression (6.06) reduces to the following:

$$f(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{[y-t+yg(\beta y)]}} \exp\left\{i\int_\tau^y \sqrt{[u-t+ug(\beta u)]}\, \mathrm{d}u\right\} \tag{6.07}$$

The latter coincides with (3.07) if we put in that equation, $C=1$ and take for τ the root of the equation (6.04). By means of the formula (6.06) we have verified that *the same* solution of equation (3.06) will have, for finite

values of y, the expression (5.06), and for large values of y, the expression (6.07).

In evaluating the integral

$$V(x, y, q) = e^{i\frac{3\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_{\Gamma} e^{ixt} \frac{f(y, t)}{\left(\frac{\partial f}{\partial y}+qf\right)_0} dt \tag{6.08}$$

we can now use both expressions (5.06) and (6.07) at the same time, namely, we can insert the expression (6.07) in the numerator and expression (5.06) in the denominator. In doing this we can simplify somewhat both expressions. Neglecting small corrections, instead of (5.06), we shall write simply

$$f(y, t) = w(t-y) \tag{6.09}$$

Also, in the formula (6.07) in the coefficient before the exponential function we shall neglect the quantity t as compared with y, and replace the exponent by the approximate expression

$$\int_{\tau}^{y} \sqrt{[u-t+ug(\beta u)]}\, du = \int_{0}^{y} \sqrt{[u+ug(\beta u)]}\, du - \frac{1}{2} t \int_{0}^{y} \frac{du}{\sqrt{[u+ug(\beta u)]}} \tag{6.10}$$

Using the notation of (4.18) and (4.20) we can write

$$f(y, t) = \sqrt{\left(2\frac{\partial p}{\partial y}\right)} e^{i\frac{\pi}{4}} e^{i\omega_0(y)-it(x+p)} \tag{6.11}$$

We are thus replacing the function $f(y, t)$ in the denominator by the Airy function, and in the numerator by the exponential function.

Substituting (6.09) and (6.11) in the integral (6.08) we get

$$V(x, y, q) = e^{i\omega_0(y)} \sqrt{\left(2x\frac{\partial p}{\partial y}\right)} \cdot \frac{1}{\sqrt{\pi}} \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t)-qw(t)} \tag{6.12}$$

The remaining integral can be considered as a known function. In Chapter 10, formula (6.20), it is denoted by

$$V_1(-p, q) = \frac{1}{\sqrt{\pi}} \cdot \int_{\Gamma} e^{-ipt} \frac{dt}{w'(t)-qw(t)} \tag{6.13}$$

and investigated in detail (see also Ref. 22). For the cases $q=0$ and $q=\sqrt{i}$ there are tables.†

Formula (6.12) gives the attenuation factor for the region near the

† The tables for $q=0$ are given in Chapter 2.

horizon. It is interesting to compare this formula with the formula (4.09) valid in the region where geometrical optics is applicable. Using (4.19) we write the expression (4.09) in the form

$$V = e^{i\omega_0(y)} \sqrt{\left(2x\frac{\partial p}{\partial y}\right)} \cdot \frac{2p}{p-iq} \cdot e^{\frac{i}{3}p^3} \tag{6.14}$$

But in Chapter 10 it was shown that the function (6.13) has, for large positive values of p, the asymptotic expression

$$V_1(-p, q) = \frac{2p}{p-iq} e^{\frac{i}{3}p^3} \tag{6.15}$$

(formula (6.24) of Chapter 10). Thus our formula (6.12) goes over in the region of direct visibility to the formula of geometrical optics.

For negative values of p the expression for $V_1(-p, q)$ can be written in the form

$$V_1(-p, q) = i2\sqrt{\pi} \sum_{n=1}^{n=\infty} \frac{e^{-ipt_n}}{(t_n-q^2)w(t_n)} \tag{6.16}$$

When $|p|$ is large ($p<0$) this series reduces to the first term which gives the exponential attenuation of the wave in the region of the shadow.

The function $V_1(-p, q)$ was first introduced in our papers on the diffraction by a body of arbitrary shape (see Chapters 1, 2 and 5). There the principle of the local field in the penumbra region was established and it was shown that in this region the field is expressed by the function $V_1(-p, q)$ which has a universal character.

The comparison of the formulae (6.12) and (6.14) permits us to say in a certain sense that the wave reaches the horizon with amplitude and phase corresponding to the laws of geometrical optics and at the horizon it suffers diffraction according to the law of the local field in the penumbra region.

This picture is found to be in complete agreement with the ideas of L. I. Mandelstam who put forward the view that in the propagation of electromagnetic waves along the surface of the earth the properties of the ground are essential not along the entire trajectory of the ray, but only in that region of the earth where the transmitter or the receiver is located ("line of departure" and "line of arrival" area).

If we accept this picture then the solution obtained in this section may be applied to that case where the properties of the earth's surface in different areas are different, provided the complex parameter q in the function $V_1(-p, q)$ corresponds to the properties of the ground in that area where the ray touches the earth.

CHAPTER 14

THEORY OF RADIOWAVE PROPAGATION IN AN INHOMOGENEOUS (STRATIFIED) ATMOSPHERE FOR A RAISED SOURCE†

Abstract. — In Section 1 of the present paper the basic equations and the boundary conditions of the problem are stated. In Section 2 the approximate form of the equations (Leontovich's parabolic equation) is considered with the corresponding boundary conditions and the conditions determining the singularity. In Section 3 the analogy between the problem formulated and the non-stationary problem of quantum mechanics is pointed out. After transformation to non-dimensional quantities (Section 4), the properties of the particular solutions of the differential equations are studied (Section 5). With the help of these particular solutions a general solution in the form of a contour integral and of a series is then constructed (Section 6). The general theory is applied to the case of super-refraction (Section 7), and an example is considered in which the curve of the modified refractive index is assumed to be composed of two rectilinear segments. In the last section (Section 8) approximate formulae for the determination of the attenuation coefficients and the height factors are derived; these formulae are similar to the semi-classical formulae of quantum mechanics. Questions concerning numerical computation methods are not discussed in this paper.

INTRODUCTION

The theory of radiowave propagation in an atmosphere with a dielectric constant dependent on height was developed in Chapter 13 for the case when the source is a vertical electric dipole situated on the earth's surface. The case of a raised source (horizontal and vertical electric and magnetic dipoles) has been considered in Chapter 12 assuming a homogeneous atmosphere. In the present paper the "combined" case of a raised source in an inhomogeneous atmosphere is investigated.

The formulae derived in Chapter 13 for the general case of an arbitrary dependence of the refractive index on the height were developed there in more detail for the case of normal refraction; then the radiowave propa-

† Fock, 1950.

gation has the same qualitative character as in a homogeneous atmosphere. The case of super-refraction, when the lower layer of the atmosphere acquires the character of a wave guide, is of independent interest and deserves special consideration. In the present paper we consider this case in some detail. For its qualitative description the analogy with the non-stationary quantum-mechanical problem of the dispersion of a wave packet in a given force field appears to be useful: it seems that this analogy has not been observed until now.

1. *Fundamental Equations and Boundary Conditions*

Let us denote by r, ϑ, φ spherical coordinates with origin at the centre of the earth's globe and with the polar axis running through the radiating dipole. We denote the earth's radius by a. We suppose the dipole to be situated at a height $h' = b-a$ above the earth's surface so that its coordinates are $r=b$, $\vartheta=0$. The dielectric constant of the air, ε, is assumed to be a function of the height $h = r-a$ above the earth's surface.

The field in air can be expressed in terms of the Debye potentials u, v according to the well-known formulae

$$\left.\begin{aligned}
E_r &= \frac{1}{r}\Delta^* u \\
E_\vartheta &= -\frac{1}{\varepsilon r}\cdot\frac{\partial^2(\varepsilon r u)}{\partial r\,\partial\vartheta}+\frac{i\omega}{c\sin\vartheta}\frac{\partial v}{\partial\varphi} \\
E_\varphi &= -\frac{1}{\varepsilon r\sin\vartheta}\cdot\frac{\partial^2(\varepsilon r u)}{\partial r\,\partial\varphi}-\frac{i\omega}{c}\cdot\frac{\partial v}{\partial\vartheta}
\end{aligned}\right\} \tag{1.01}$$

$$\left.\begin{aligned}
H_r &= -\frac{1}{r}\Delta^* v \\
H_\vartheta &= \frac{i\omega}{c}\cdot\frac{\varepsilon}{\sin\vartheta}\cdot\frac{\partial u}{\partial\varphi}+\frac{1}{r}\frac{\partial^2(rv)}{\partial r\,\partial\vartheta} \\
H_\varphi &= -\frac{i\omega}{c}\,\varepsilon\frac{\partial u}{\partial\vartheta}+\frac{1}{r\sin\vartheta}\cdot\frac{\partial^2(rv)}{\partial r\,\partial\varphi}
\end{aligned}\right\} \tag{1.02}$$

The same expressions are valid for the field within the earth if we understand by ε the complex dielectric constant of the earth. The dependence on time is assumed to be of the form $e^{-i\omega t}$. The symbol Δ^* in (1.01) and (1.02) denotes the spherical Laplace operator

$$\Delta^* u = \frac{1}{\sin\vartheta}\cdot\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial u}{\partial\vartheta}\right)+\frac{1}{\sin^2\vartheta}\frac{\partial^2 u}{\partial\varphi^2} \tag{1.03}$$

Maxwell's equations will be satisfied if the functions u and v satisfy the equations

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}\left(\frac{1}{\varepsilon}\cdot\frac{\partial(\varepsilon r u)}{\partial r}\right)+\frac{\Delta^* u}{r^2}+\frac{\omega^2}{c^2}\,\varepsilon u=0 \tag{1.04}$$

and

$$\frac{1}{r}\cdot\frac{\partial^2(rv)}{\partial r^2}+\frac{\Delta^* v}{r^2}+\frac{\omega^2}{c^2}\,\varepsilon v=0 \tag{1.05}$$

The continuity of the tangential components of the field will be secured if the quantities

$$\varepsilon r u;\ \frac{1}{\varepsilon}\cdot\frac{\partial(\varepsilon r u)}{\partial r};\qquad rv;\ \frac{\partial(rv)}{\partial r}, \tag{1.06}$$

are continuous at $r=a$. From continuity conditions the approximate form of the boundary conditions (Leontovich conditions) can be obtained by well-known reasoning. If we put $k=\omega/c$, denote the complex dielectric constant of the earth by η and retain the notation ε for the dielectric constant of air, then we have

$$\frac{\partial(\varepsilon r u)}{\partial r}=-ik\frac{\varepsilon}{\sqrt{\eta}}(\varepsilon r u)\qquad(\text{for } r=a) \tag{1.07}$$

and

$$\frac{\partial(rv)}{\partial r}=-ik\sqrt{\eta}\cdot(rv)\qquad(\text{for } r=a) \tag{1.08}$$

In the following the field for which $u\neq 0, v=0$ will be called "vertically polarized" and the field for which $u=0$, $v\neq 0$ – "horizontally polarized". In this sense, the field of a vertical electric dipole remains vertically polarized in all the space. The field of a vertical magnetic dipole (horizontal loop) has horizontal polarization everywhere. A horizontal electric dipole, however, excites fields of both forms: horizontally and vertically polarized. In the case of a homogeneous atmosphere, the vertically polarized field decreases with increasing distance more slowly than the horizontally polarized. Consequently, the field from a horizontal electric dipole at small distances from the source will have a predominant horizontal polarization, but at large distances (in the region far beyond the horizon) the polarization will be predominantly vertical.

The vertically polarized field can be expressed in terms of the function U (the Hertz function of a vertical electric dipole) which has the following properties. The function U satisfies the same differential equation (1.04) and the same boundary conditions (1.07) as u and has, near the source,

a singularity of the form

$$U = \frac{e^{ikR}}{R} + U^* \tag{1.09}$$

where U^* remains finite,

$$R = \sqrt{[r^2+b^2-2rb\cos\vartheta]}, \qquad \text{and} \qquad k = \omega/c \tag{1.10}$$

Similarly, the horizontally polarized field can be expressed in terms of the function W (the Hertz function of a vertical magnetic dipole) which satisfies the same differential equation (1.05) and the same boundary conditions (1.08) as v and has near the source a singularity of the form

$$W = \frac{e^{ikR}}{R} + W^* \tag{1.11}$$

where W^* remains finite.

The fields of the vertical and horizontal electric and magnetic dipoles with moment M are expressed through the functions U and W defined above.

For the vertical electric dipole we have to put

$$u = \frac{M}{b} U, \quad v = 0 \tag{1.12}$$

For the vertical magnetic dipole (horizontal loop) we have

$$u = 0; \qquad v = \frac{M}{b} W \tag{1.13}$$

For the horizontal electric dipole directed along the x-axis, the functions u and v in (1.01) and (1.02) are determined from the equations

$$\left.\begin{aligned} \Delta^* u &= -M\frac{\partial}{\partial\vartheta}\left(\frac{\partial U}{\partial b}+\frac{U}{b}\right)\cos\varphi \\ \Delta^* v &= -ikM\frac{\partial W}{\partial\vartheta}\sin\varphi \end{aligned}\right\} \tag{1.14}$$

Finally, for the horizontal magnetic dipole directed along the x-axis we have

$$\left.\begin{aligned} \Delta^* u &= -ikM\frac{\partial U}{\partial\vartheta}\sin\varphi \\ \Delta^* v &= M\frac{\partial}{\partial\vartheta}\left(\frac{\partial W}{\partial b}+\frac{W}{b}\right)\cos\varphi \end{aligned}\right\} \tag{1.15}$$

Thus, in all four cases the study of the field reduces to the study of the functions U and W.

2. *Approximate Form of the Equations*

Turning to the approximate form of the equations, we denote by ε_1 the value of the dielectric constant of air near the source (in practice we can put $\varepsilon_1 = 1$) and we put

$$s = a\vartheta \tag{2.01}$$

so that s is the horizontal distance between the source and the observation point, measured along the arc of the earth's surface.

Instead of U and W we introduce the slowly varying functions U_2 and W_2 putting

$$U = \frac{\varepsilon_1 \, e^{iks}}{\varepsilon r \sqrt{(\sin\vartheta)}} U_2 \tag{2.02}$$

and

$$W = \frac{e^{iks}}{r\sqrt{(\sin\vartheta)}} W_2 \tag{2.03}$$

As shown in Chapter 13, if small quantities are neglected the equation for U_2 becomes

$$\frac{\partial^2 U_2}{\partial h^2} + 2ik\frac{\partial U_2}{\partial s} + k^2\left(\varepsilon - 1 + \frac{2h}{a}\right) U_2 = 0 \tag{2.04}$$

In this equation, the quantities h (height) and s (horizontal distance) are taken as independent variables instead of r and ϑ. In our approximation, the equation for W_2 will have the same form, namely

$$\frac{\partial^2 W_2}{\partial h^2} + 2ik\frac{\partial W_2}{\partial s} + k^2\left(\varepsilon - 1 + \frac{2h}{a}\right) W_2 = 0 \tag{2.05}$$

The equations (2.04) and (2.05) will be called the Leontovich parabolic equations.

In the boundary conditions on the earth's surface ($h = 0$) we can neglect the difference between the dielectric constant in air and unity.

On the other hand, we can improve somewhat these conditions by using the results we obtained by the series summation method (see Chapters 12 and 6). This improvement consists in replacing η by $\eta+1$ in equation (1.07), and by $\eta-1$ in equation (1.08). We thus obtain from (1.07) and (1.08)

$$\frac{\partial U_2}{\partial h} = -\frac{ik}{\sqrt{(\eta+1)}} U_2 \qquad (\text{for } h = 0) \tag{2.06}$$

and

$$\frac{\partial W_2}{\partial h} = - ik\sqrt{(\eta-1)}W_2 \qquad (\text{for } h = 0) \tag{2.07}$$

We have further to formulate the requirement that, in the region not too far from the source where the curvature of the earth's surface and that of the rays can be neglected, the reflection formula for the plane earth must hold. If the height of the source above the earth is $h' = b-a$ then this requirement means that in the aforementioned region:

$$U_2 = \sqrt{\left(\frac{a}{s}\right)}\left[e^{i\frac{k(h-h')^2}{2s}} + e^{i\frac{k(h+h')^2}{2s}} \cdot \frac{h+h'-\dfrac{s}{\sqrt{(\eta+1)}}}{h+h'+\dfrac{s}{\sqrt{(\eta+1)}}}\right] \tag{2.08}$$

and

$$W_2 = \sqrt{\left(\frac{a}{s}\right)}\left[e^{i\frac{k(h-h')^2}{2s}} + e^{i\frac{k(h+h')^2}{2s}} \cdot \frac{h+h'-s\sqrt{(\eta-1)}}{h+h'+s\sqrt{(\eta-1)}}\right] \tag{2.09}$$

The second exponential in (2.08) and (2.09) is multiplied by factors representing the approximate values of the Fresnel coefficients for vertical and horizontal polarization. These two formulae are generalizations of formula (1.28) of Chapter 13.

It is to be noted that the expressions (2.08) and (2.09) themselves satisfy approximately the boundary conditions (2.06) and (2.07).

In the case of a field above a perfectly conducting surface ($\eta = \infty$) the boundary conditions (2.06) and (2.07) become

$$\frac{\partial U_2}{\partial h} = 0 \qquad (\text{for } h = 0) \tag{2.10}$$

and

$$W_2 = 0 \qquad (\text{for } h = 0) \tag{2.11}$$

and the reflection formulae can be written as follows:

$$U_2 = \sqrt{\left(\frac{a}{s}\right)}\left[e^{\frac{ik(h-h')^2}{2s}} + e^{\frac{ik(h+h')^2}{2s}}\right] \tag{2.12}$$

and

$$W_2 = \sqrt{\left(\frac{a}{s}\right)}\left[e^{i\frac{k(h-h')^2}{2s}} - e^{\frac{ik(h+h')^2}{2s}}\right] \tag{2.13}$$

3. *Analogy With the Non-Stationary Problem of Quantum Mechanics*

In the preceding section we formulated the problem of wave propagation in a spherical layer with variable refractive index. This problem is analogous to the quantum-mechanical problem of the motion of a wave packet in a given field of force.

Let us write Schrödinger's equation for the motion of a particle of mass m_0 in a force field with potential energy Φ. Denoting the coordinate of the particle by x, the time by t, Planck's constant (divided by 2π) by $\hbar$ we have

$$\frac{\partial^2\psi}{\partial x^2}+2i\frac{m_0}{\hbar}\frac{\partial\psi}{\partial t}-\frac{2m_0}{\hbar^2}\Phi\psi = 0 \tag{3.01}$$

Comparing Schrödinger's equation (3.01) with the Leontovich equations (2.04) or (2.05) for U_2 and W_2, we see that these equations are of the same form, provided the following correspondence between the respective quantities is established: the coordinate x is proportional to the height h, the time t is proportional to the horizontal distance s and the potential energy Φ is proportional to the negative of the quantity $(\varepsilon-1+2h/a)$ which differs from the so-called reduced (or modified) refractive index

$$M = 10^6\left(\frac{\varepsilon-1}{2}+\frac{h}{a}\right) = 10^6\left(n-1+\frac{h}{a}\right) \tag{3.02}$$

only by a constant factor.

Thus, the Leontovich parabolic equation for the wave amplitude coincides in form with the non-stationary Schrödinger equation.

The analogy between the two problems is not limited to the form of the differential equations but extends also to the boundary and "initial" conditions.

In quantum mechanics, self-adjoint differential equations and boundary conditions are considered. This corresponds in the electromagnetic problem to the case when there is no absorption in air and in the earth (when the refractive index of air is real and the earth is a perfect conductor). This case is most interesting for the super-refraction problem as well. The quantum-mechanical methods can also be generalized to the case when absorption is present.

If the earth is a perfect conductor, then the boundary conditions for U_2 and W_2 take the form (2.10) and (2.11) and the corresponding conditions of the quantum-mechanical problem are

$$\frac{\partial\psi}{\partial x} = 0 \qquad (\text{for } x = 0) \tag{3.03}$$

or

$$\psi = 0 \qquad (\text{for } x = 0) \tag{3.04}$$

The initial conditions consist in prescribing the initial value of the wave function

$$\psi = \psi_0(x) \qquad (\text{for } t = 0, \quad 0 < x < \infty) \tag{3.05}$$

The function ψ which satisfies the differential equation, the initial and boundary conditions can be sought in the form

$$\psi(x, t) = \int_0^\infty F(x, x', t)\psi_0(x')\,\mathrm{d}x' \tag{3.06}$$

The function F must satisfy for all values of x' the differential equation

$$\frac{\partial^2 F}{\partial x^2} + 2i\frac{m_0}{\hbar}\frac{\partial F}{\partial t} - \frac{2m_0}{\hbar^2}\Phi F = 0 \tag{3.07}$$

and the boundary condition of the form (3.03) or (3.04) (the same as ψ). In order that the expression (3.06) should reduce at $t=0$ to $\psi_0(x)$ the function F must have at $t\to 0$ a singularity depending on the kind of the boundary condition. In case of the condition

$$\frac{\partial F}{\partial x} = 0 \qquad (\text{for } x = 0) \tag{3.08}$$

the singularity of F must have the form

$$F(x, x', t) = \mathrm{e}^{-i\frac{\pi}{4}}\sqrt{\left(\frac{m_0}{2\pi\hbar t}\right)}\left[\mathrm{e}^{i\frac{m_0(x-x')^2}{2\hbar t}} + \mathrm{e}^{i\frac{m_0(x+x')^2}{2\hbar t}}\right] \tag{3.09}$$

In case of the condition

$$F = 0 \qquad (\text{for } x = 0) \tag{3.10}$$

the singularity of F must have the form

$$F(x, x', t) = \mathrm{e}^{-i\frac{\pi}{4}}\sqrt{\left(\frac{m_0}{2\pi\hbar t}\right)}\left[\mathrm{e}^{i\frac{m_0(x-x')^2}{2\hbar t}} - e^{i\frac{m_0(x+x')^2}{2\hbar t}}\right] \tag{3.11}$$

Comparing these formulae with (2.12) and (2.13) we see that, for corresponding boundary conditions, the singularity of F exactly coincides with the singularity of U_2 and W_2. Indeed, equating the height h to the coordinate x we must put

$$h = x; \qquad h' = x'; \qquad \frac{s}{k} = \frac{\hbar t}{m_0} \tag{3.12}$$

Expressing F in terms of the variables h, h' and s we will have in case of the boundary condition (3.08)

$$F = F_2(h, h', s) \tag{3.13}$$

The function F_2 satisfies the same equation as U_2, the boundary condition

$$\frac{\partial F_2}{\partial h} = 0 \qquad (\text{for } h = 0) \tag{3.14}$$

and has the singularity

$$F_2(h, h', s) = \mathrm{e}^{-i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi s}\right)} \left[\mathrm{e}^{i\frac{k(h-h')^2}{2s}} + \mathrm{e}^{i\frac{k(h+h')^2}{2s}}\right] \tag{3.15}$$

In the case of the boundary condition (3.10) we put

$$F = G_2(h, h', s) \tag{3.16}$$

where G_2 satisfies the same differential equation as W_2, the boundary condition

$$G = 0 \qquad (\text{for } h = 0) \tag{3.17}$$

and has the singularity

$$G_2(h, h', s) = \mathrm{e}^{-i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi s}\right)} \left[\mathrm{e}^{i\frac{k(h-h')^2}{2s}} - \mathrm{e}^{i\frac{k(h+h')^2}{2s}}\right] \tag{3.18}$$

We see that F_2 differs from U_2 only by a constant factor, as does G_2 from W_2. We have, namely,

$$U_2 = \mathrm{e}^{i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi a}\right)} F_2 \tag{3.19}$$

and

$$W_2 = \mathrm{e}^{i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi a}\right)} G_2 \tag{3.20}$$

If we denote by $f(h, s)$ a function which satisfies the same equation and the same boundary conditions as U_2 and which takes, for $s=0$, the value

$$f(h, s) = f_0(h) \qquad (\text{for } s = 0) \tag{3.21}$$

then we can write, using (3.06) and (3.19)

$$f(h, s) = \mathrm{e}^{-i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi a}\right)} \int_0^\infty U_2(h, h', s) f_0(h')\, \mathrm{d}h' \tag{3.22}$$

Similarly, if $f(h, s)$ satisfies the same boundary conditions as W_2 then

$$f(h, s) = \mathrm{e}^{-i\frac{\pi}{4}} \sqrt{\left(\frac{k}{2\pi a}\right)} \int_0^\infty W_2(h, h', s) f_0(h')\, \mathrm{d}h' \tag{3.23}$$

The last two formulae are valid not only for boundary conditions corresponding to a perfectly conducting earth (when there is an analogy with quantum mechanics) but even for the more general boundary conditions (2.06) and (2.07), the singularities of U_2 and W_2 being given by the formulae (2.08) and (2.09).

If the function $f_0(h)$ is different from zero only in the vicinity of the point $h=h'$ and such that the integral of f_0 over this region is finite, then $f(h, s)$ defined by (3.22) or (3.23) will be, for not too small s, proportional to U_2 or W_2, respectively. Thus the functions U_2 and W_2 correspond to a point source at the height h', as they ought.

In quantum-mechanical language, it can be said that the function ψ, proportional to U_2 or to W_2 gives the solution of the problem of the dispersion of a wave packet originally concentrated in the vicinity of one point.

It is known from quantum-mechanics, that the speed of dispersion depends on the form of the potential energy. Let us imagine that the motion of the particle is bounded on one side by an impermeable wall. If the potential energy is such that the force is always directed from the wall, then dispersion proceeds rapidly. If, however, there is a force that keeps the particles in some region, where the potential energy has a minimum, or near the wall, then the dispersion proceeds slowly or not at all. In this case Schrödinger's equation admits solutions corresponding to stationary or almost stationary states.

The wave function of an almost-stationary state is different from zero at the initial moment only in the region of minimum potential energy. In the course of time, the amplitude of the wave function in this region decreases, and disintegration of the initial almost-stationary state takes place. The decrease in the amplitude proceeds exponentially and the rapidity of disintegration is characterized by the coefficient in the exponent, which is called the disintegration constant.

If the initial wave function itself is not a wave function of an almost-stationary state, then in its expansion the term corresponding to the almost-stationary state can be separated out and for large values of time this will be the principal term.

In our electromagnetic problem, the horizontal distance s plays the part of the time t of the quantum mechanical problem. To the dispersion of the wave packet corresponds the decrease of the field amplitude with increasing horizontal distance. The earth's surface ($h=0$) plays the part of the wall. The wall will be impermeable if the earth is an absolute conductor; for finite conductivity the wall will be absorbent and a decrease of

amplitude will be due not only to the leakage of waves into the upper layers of the atmosphere but also to the absorption by the earth. The part of the potential energy is played as stated above by the modified refractive index, M, taken with the opposite sign. The dependence of the modified refractive index on height is shown on Fig. 1.

The solid curve shows the dependence of M on the height if superrefraction is present. The dotted continuation of the rectilinear part of the curve

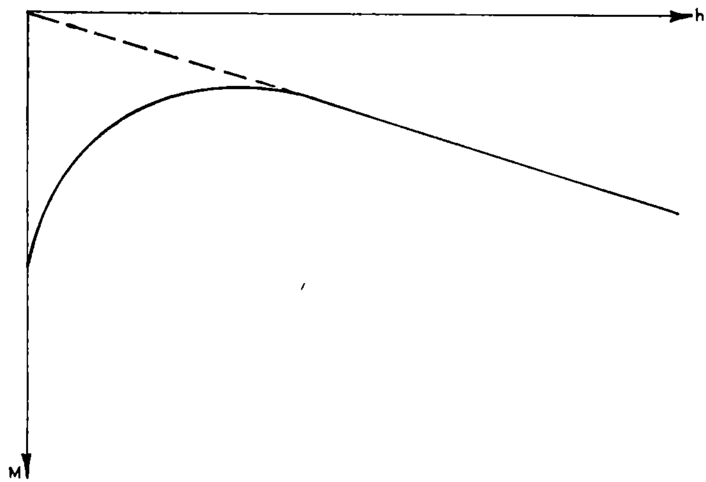

FIG. 1.

corresponds to the case when there is no superrefraction and one can introduce the "equivalent radius" of the earth which is proportional to the angular coefficient of the straight part of the curve relative to the M axis.

If the curve in Fig. 1 is considered as the potential energy curve, then it will be clear that the presence of the maximum for $(-M)$ (minimum for M) which is characteristic for superrefraction, is a necessary condition for the existence of an almost-stationary state. In fact, if we denote the height corresponding to the maximum of potential energy (to the "barrier"), by h_m then in the region $h < h_m$ the force will be squeezing the wave packet to the wall and not letting it go out into the $h > h_m$ region.

Now in our electromagnetic problem, the presence of an almost-stationary state means that the amplitude of the wave decreases with increasing distance abnormally slowly, so that its attenuation coefficient (corres-

ponding to the constant of disintegration) is abnormally small. Hence it follows that the condition for the existence of the almost-stationary state is the same as the condition for the possibility of long-range propagation of radio waves.

The analogy with quantum mechanics, which was exposed here, allows us to build up a qualitative picture of the phenomenon of long-range radiowave propagation. This analogy offers also the possibility of applying some mathematical methods used in quantum mechanics to problems of radiophysics. On the other hand, our methods of solving problems of radiowave propagation can be applied in quantum mechanics†. However, this question lies beyond the scope of this paper.

4. *Transformation to Non-dimensional Quantities*

Let us return to the solution of the problem formulated in Section 2. We have to determine the functions U_2 and W_2 which satisfy the differential equations (2.04) and (2.05), the boundary conditions (2.06) and (2.07) and the conditions (2.08) and (2.09) characterizing the singularity. This problem was solved, in previous chapters, for two cases: (*a*) inhomogeneous (stratified) atmosphere, source on the earth, and (b) homogeneous atmosphere, raised source. Now we will show that this problem can be solved for the general case of stratified atmosphere and raised source.

Let us introduce into our equations the non-dimensional quantities used in previous chapters. We consider the coefficient of U_2 in equation (2.04). This coefficient is proportional to the quantity

$$\frac{\varepsilon-1}{2}+\frac{h}{a} = 10^{-6}\, M(h) \tag{4.01}$$

where $M(h)$ is the modified refractive index. We suppose that, beginning with some height $h=H$, this quantity can be approximated by a linear function of h and we put

$$\frac{\varepsilon-1}{2}+\frac{h}{a} = \alpha+\frac{h}{a^*} \tag{4.02}$$

where a^* is the so-called equivalent radius of the earth and α is a small constant (for example, $\alpha < 0{\cdot}0005$). In the simplest case, it can be assumed that $\varepsilon=1$ for $h > H$ then we have to put $\alpha=0$ and $a^*=a$.

In the region where the formula (4.02) is valid, the equation for U_2 takes the form

$$\frac{\partial^2 U_2}{\partial h^2} = 2ik\frac{\partial U_2}{\partial s}+k^2\left(2\alpha+\frac{2h}{a^*}\right)U_2 = 0 \tag{4.03}$$

† Thus, the modern theory of complex angular momentum is related to our summation methods using integration in the complex ν-plane. (Note added in the present edition).

In order to eliminate the constant α in the last term, we make the substitution

$$U_2 = C\,e^{i\alpha ks}\,\Psi \tag{4.04}$$

where C is a constant to be disposed of later. Then equation (4.03) reduces to the form

$$\frac{\partial^2\Psi}{\partial h^2}+2ik\frac{\partial\Psi}{\partial s}+k^2\frac{2h}{a^*}\Psi = 0 \tag{4.05}$$

Introducing the abbreviation

$$m = \left(\frac{ka^*}{2}\right)^{1/3} \tag{4.06}$$

and putting

$$ks = 2m^2x; \qquad kh = my; \qquad kh' = my' \tag{4.07}$$

we can write equation (4.05) in the form

$$\frac{\partial^2\Psi}{\partial y^2}+i\frac{\partial\Psi}{\partial x}+y\Psi = 0 \tag{4.08}$$

The same substitutions reduce the more exact equation (2.04) to the form

$$\frac{\partial^2\Psi}{\partial y^2}+i\frac{\partial\Psi}{\partial x}+[y+r(y)]\,\Psi = 0 \tag{4.09}$$

where

$$r(y) = m^2\left(\varepsilon-1+\frac{2h}{a}-2\alpha-\frac{2h}{a^*}\right) \tag{4.10}$$

The quantity $r(y)$ characterizes the anomalous behaviour of the refractive index near the earth's surface. From a certain value of y upwards, the quantity $r(y)$ can be set equal to zero. If we put $\alpha=0$ and $a^*=a$, we will simply have

$$r(y) = m^2(\varepsilon-1) \tag{4.11}$$

We have also to express the boundary conditions in the new variables and the conditions characterizing the singularity. Setting

$$q = \frac{im}{\sqrt{(\eta+1)}} \tag{4.12}$$

we have

$$\frac{\partial\Psi}{\partial y}+q\Psi = 0 \qquad (\text{for } y = 0) \tag{4.13}$$

The constant C in (4.04) can be chosen so that the equation corresponding to (3.22) assumes the form

$$f(x, y) = \int_0^\infty \Psi(x, y, y') f_0(y')\, dy' \tag{4.14}$$

Then the equation defining the singularity of Ψ becomes

$$\Psi = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{(\pi x)}} \left[e^{i\frac{(y-y')^2}{4x}} + e^{i\frac{(y+y')^2}{4x}} \cdot \frac{y+y'+2iqx}{y+y'-2iqx} \right] \tag{4.15}$$

Comparison of (2.08) with (4.15) gives

$$C = e^{i\frac{\pi}{4}} \cdot \frac{\sqrt{(2\pi ka)}}{m} \tag{4.16}$$

The function W_2 differs from U_2 only in that the quantity $\dfrac{1}{\sqrt{(\eta+1)}}$ involved in the boundary conditions and in the equation defining the singularity is replaced by the quantity $\sqrt{(\eta-1)}$. This corresponds to replacing q by

$$q_1 = im\sqrt{(\eta-1)} \tag{4.17}$$

In practice, it is possible to put $q_1 = \infty$ in all cases.

Along with the function Ψ we will consider the function

$$V(x, y, y', q) = 2\sqrt{(\pi x)} \cdot e^{i\frac{\pi}{4}}\, \Psi \tag{4.18}$$

which will be called the attenuation factor. The quantities U_2 and W_2 are expressed in terms of V as follows

$$U_2 = e^{i\alpha ks} \sqrt{\left(\frac{a}{s}\right)}\, V \tag{4.19}$$

and

$$U = \frac{e^{i(1+\alpha)ks}}{\sqrt{(sa \sin s/a)}}\, V(x, y, y', q) \tag{4.20}$$

The function W is obtained from (4.20) if q is replaced by q_1.

5. *Properties of the Particular Solutions*

In order to construct the function Ψ satisfying the formulated conditions, it is necessary to investigate the particular solutions of the equation (4.09) which are obtained by separating the variables in (4.09). Putting

$$\Psi = e^{ixt} f(y, t) \tag{5.01}$$

we obtain for $f(y, t)$ the equation

$$\frac{d^2 f}{dy^2}+[y+r(y)-t]f = 0 \tag{5.02}$$

We denote by $f^0(y, t)$ and $f^*(y, t)$, the solutions of equation (5.02) which satisfy the initial conditions

$$f^0(0, t) = 1; \qquad \left(\frac{\partial f^0}{\partial y}\right)_{y=0} = 0 \tag{5.03}$$

and

$$f^*(0, t) = 0; \qquad \left(\frac{\partial f^*}{\partial y}\right)_{y=0} = 1 \tag{5.04}$$

The general solution of equation (5.02) will have the form

$$f(y, t) = A^0 f^0(y, t)+A^* f^*(y, t) \tag{5.05}$$

On the other hand, if the function $r(y)$ decreases rapidly enough as y increases, then for real t equation (5.02) will have one integral (determined apart from a constant factor independent of y) which behaves at large y as $w_1(t-y)$, and another integral which behaves as $w_2(t-y)$ where w_1 and w_2 are complex Airy functions. These functions represent solutions of the equation

$$\frac{d^2 w}{dy^2}+(y-t)w = 0 \tag{5.06}$$

obtained from (5.02) by replacing $r(y)$ by zero. The functions w_1 and w_2 have the asymptotic expressions

$$w_1(t-y) = e^{i\frac{\pi}{4}}(y-t)^{-\frac{1}{4}}e^{i\frac{2}{3}(y-t)^{3/2}} \tag{5.07}$$

and

$$w_2(t-y) = e^{-i\frac{\pi}{4}}(y-t)^{-\frac{1}{4}}e^{-i\frac{2}{3}(y-t)^{3/2}} \tag{5.08}$$

Consequently, the behaviour of the general integral of equation (5.02) for $y\to\infty$ and for real t can be characterized by the constants C_1 and C_2 in the expression

$$f(y, t) = C_1 w_1(t-y)+C_2 w_2(t-y) \tag{5.09}$$

Let us establish the connection between the constants A^0, A^*, C_1 and C_2 (which can be functions of the parameter t).

By virtue of equations (5.02) and (5.06), we have

$$\frac{d}{dy}\left(w\frac{df}{dy}-f\frac{dw}{dy}\right) = -r(y)\cdot f\cdot w(t-y) \tag{5.10}$$

In this equality, we can put successively $w=w_1$ then $w=w_2$ and integrate it between 0 and ∞. Using the relation

$$\frac{\partial w_1}{\partial y}w_2-\frac{\partial w_2}{\partial y}w_1 = 2i \tag{5.11}$$

we have

$$\lim_{y\to\infty}\left(w_2\frac{df}{dy}-f\frac{dw_2}{dy}\right) = 2iC_1 \tag{5.12}$$

and

$$\lim_{y\to\infty}\left(w_1\frac{df}{dy}-f\frac{dw_1}{dy}\right) = -2iC_2 \tag{5.13}$$

After integration, the equality (5.10) yields

$$2iC_1 = A^0w_2'(t)+A^*w_2(t)-\int_0^\infty r(y)f(y,t)w_2(t-y)\,dy \tag{5.14}$$

and

$$-2iC_2 = A^0w_1'(t)+A^*w_1(t)-\int_0^\infty r(y)f(y,t)w_1(t-y)\,dy \tag{5.15}$$

If, here, we insert the expression (5.05) for $f(y, t)$ we obtain the desired connection between the constants A^0, A^*, C_1 and C_2 in the form

$$\begin{aligned}2iC_1 = A^0&\left[w_2'(t)-\int_0^\infty r(y)f^0(y,t)w_2(t-y)\,dy\right]+\\ &+A^*\left[w_2(t)-\int_0^\infty r(y)f^*(y,t)w_2(t-y)\,dy\right]\end{aligned} \tag{5.16}$$

and

$$\begin{aligned}-2iC_2 = A^0&\left[w_1'(t)-\int_0^\infty r(y)f^0(y,t)w_1(t-y)\,dy\right]+\\ &+A^*\left[w_1(t)-\int_0^\infty r(y)f^*(y,t)w_1(t-y)\,dy\right]\end{aligned} \tag{5.17}$$

We observe that the coefficients of A^0 and A^* in these equations are integral transcendental functions of t. Indeed, the functions $f^0(y, t)$ and $f^*(y, t)$ are integral functions of t, and the integration range is only formally infinite. In practice the integral is taken between finite limits since $r(y)$ can be set equal to zero starting from a certain y upwards. (The same conclusion will also hold without this limitation on $r(y)$ provided $r(y)$ decreases rapidly enough at infinity.)

From this it follows that if the constants A^0 and A^* are integral transcendental functions of t, then the same will be true for the constants C_1 and C_2. This enables us to apply the equations (5.16) and (5.17), derived for real values of t, to the case of arbitrary complex values of t.

If we put

$$A^0 = A_1^0(t) \equiv w_1(t) - \int_0^\infty r(y) f^*(y, t) w_1(t-y) \, \mathrm{d}y \tag{5.18}$$

and

$$A^* = A_1^*(t) \equiv -w_1'(t) + \int_0^\infty r(y) f^0(y, t) w_1(t-y) \, \mathrm{d}y \tag{5.19}$$

then $C_2 = 0$ and the expression

$$f_1(y, t) = A_1^0(t) f^0(y, t) + A_1^*(t) f^*(y, t) \tag{5.20}$$

will be that solution of (5.02) which behaves like $w_1(t-y)$ as $y \to \infty$ and which is also an integral transcendental function of t.

Similarly, if we put

$$A^0 = A_2^0(t) \equiv w_2(t) - \int_0^\infty r(y) f^*(y, t) w_2(t-y) \, \mathrm{d}y \tag{5.21}$$

and

$$A^* = A_2^*(t) \equiv -w_2'(t) + \int_0^\infty r(y) f^0(y, t) w_2(t-y) \, \mathrm{d}y \tag{5.22}$$

then we will have $C_1 = 0$ and the expression

$$f_2(y, t) = A_2^0(t) f^0(y, t) + A_2^*(t) f^*(y, t) \tag{5.23}$$

will behave as $w_2(y-t)$ for $y \to \infty$ and will be an integral function of t.

The solution $f_1(y, t)$ will have the asymptotic expression

$$f_1(y, t) = \frac{c' e^{i\frac{\pi}{4}}}{\sqrt[4]{[y-t+r(y)]}} \cdot \exp\left\{ i \int_\tau^y \sqrt{[u-t+r(u)]} \, \mathrm{d}u \right\} \tag{5.24}$$

and the solution $f_2(y, t)$ will have the asymptotic expression

$$f_2(y, t) = \frac{c'' e^{-i\frac{\pi}{4}}}{\sqrt[4]{[y-t+r(y)]}} \cdot \exp\left\{-i\int_\tau^y \sqrt{[u-t+r(u)]}\, du\right\} \tag{5.25}$$

where c', c'' and τ are constants. If we put $r(y)=0$, $\tau=t$, $c'=c''=1$ then the formulae (5.24) and (5.25) will go over into the asymptotic expressions (5.07) and (5.08) for w_1 and w_2.

We already used the solution $f_1(y, t)$ in Chapter 13 where, however, it was assumed without proof that such solution exists (namely, a solution which has the asymptotic expression (5.24) and is at the same time an integral transcendental function of t). The formulae (5.18)–(5.23), used in the proof of this statement, can also be applied to the practical (numerical) construction of this integral.

For complex values of t the function $f_1(y, t)$ increases with increasing y and the integral of the square of $f_1(y, t)$, taken over y from 0 to ∞ will diverge. However, using appropriate assumptions about the behaviour of $r(y)$ in the complex y-plane, the function $f_1(y, t)$ will behave like $w_1(t-y)$ for complex y and will tend to zero on the ray $y=r\,e^{i\alpha}$ (where $\alpha=\pi/3$), so that the integral

$$I = \int_0^{\infty e^{i\alpha}} f_1^2(y, t)\, dy \tag{5.26}$$

will converge. Let us evaluate this integral. Differentiating equation (5.02) with respect to t we have

$$\frac{d^2}{dy^2}\left(\frac{\partial f}{\partial t}\right) + [y+r(y)-t]\frac{\partial f}{\partial t} = f \tag{5.27}$$

Using this and (5.02) we obtain the relation

$$\left(f\frac{\partial^2 f}{\partial y\,\partial t} - \frac{\partial f}{\partial t}\cdot\frac{\partial f}{\partial y}\right)\Bigg|_0^y = \int_0^y f^2\, dy \tag{5.28}$$

Putting, here, $f=f_1(y, t)$ and taking $\infty e^{i\alpha}$ for the upper limit of integration we have

$$\int_0^{\infty e^{i\alpha}} f_1^2(y, t)\, dy = -\left(f_1\frac{\partial^2 f_1}{\partial y\,\partial t} - \frac{\partial f_1}{\partial t}\cdot\frac{\partial f_1}{\partial y}\right)_0 \tag{5.29}$$

6. *Construction of the Solution in Form of a Contour Integral and in Form of a Series*

In the previous section we established the existence of two solutions of the ordinary differential equation

$$\frac{d^2f}{dy^2}+[y+r(y)-t]f = 0 \tag{6.01}$$

which are integral transcendental functions of the parameter t and have the asymptotic expressions (5.24) and (5.25). These solutions, which we denoted by $f_1(y, t)$ and $f_2(y, t)$ are defined by the formulae (5.20) and (5.23).

We will show now by means of the functions f_1 and f_2 that we can construct for V and Ψ a contour integral giving the solution of our problem. Our reasoning will be similar to the reasoning in Section 3 of Chapter 13, and the final formulae will correspond to (2.24) and (3.10) of Chapter 12.

Let us denote by $D_{12}(t)$ the Wronskian

$$D_{12}(t) = f_1\frac{\partial f_2}{\partial y}-f_2\frac{\partial f_1}{\partial y} \tag{6.02}$$

We put

$$F(t, y, y', q)$$
$$= \frac{1}{D_{12}(t)}f_1(y', t)\left[f_2(y, t)-\frac{f_2'(0, t)+qf_2(0, t)}{f_1'(0, t)+qf_1(0, t)}f_1(y, t)\right] \tag{6.03}$$

where the primes of f_1 and f_2 denote derivatives with respect to y. We suppose $y' > y$, and form the integral

$$\Psi = \frac{1}{2\pi i}\int e^{ixt}F(t, y, y', q)\, dt \tag{6.04}$$

taken along a contour which encloses in a positive direction all the poles of the integrand.

From the definition of F it follows that F is a meromorphic function of the variable t (i.e. having for finite t no singularities except the poles). The function F is completely determined even if the functions f_1 and f_2, involved in it, are determined only apart from factors independent of y. Since, for all values of t, the integrals f_1 and f_2 are independent (this is seen from their asymptotic expressions), the Wronskian D_{12} has no roots and the only poles of F are the roots of the equation

$$f_1'(0, t)+qf_1(0, t) = 0 \tag{6.05}$$

If in the differential equation (6.01) the function $r(y)$ is zero we can put

$$f_1(y, t) = w_1(t-y); \qquad f_2(y, t) = w_2(t-y) \tag{6.06}$$

Then

$$D_{12}(t) = -2i \tag{6.07}$$

and the expression (6.03) for F reduces to the formula (2.21) of Chapter 12.

We now proceed to show that the function Ψ defined by the contour integral (6.04) satisfies all the conditions required.

First of all, Ψ satisfies the differential equation (4.09); this is evident since the integrand satisfies it. Further, the function Ψ satisfies the boundary condition (4.13) since we have for all t and y'

$$\frac{\partial F}{\partial y}+qF = 0 \qquad (\text{for } y = 0) \tag{6.08}$$

There remains to show that Ψ has the required singularity.

With the aid of the asymptotic expressions (5.24) and (5.25) we can show that if x, y are small and the ratio y/x is large then the principal segment of the integration path in (6.04) will lie at large negative values of t. But if t is large and negative, then the dominating term in the coefficient of f in the differential equation (6.01) will be $-t$. Therefore, for large negative t we have, approximately:

$$f_1(y, t) \sim f_1(0, t)\, e^{iy\sqrt{(-t)}} \tag{6.09}$$

and

$$f_2(y, t) \sim f_2(0, t)\, e^{-iy\sqrt{(-t)}} \tag{6.10}$$

Inserting these expressions in the formula (6.03) for F, we obtain

$$F = \frac{i}{2\sqrt{(-t)}}\left\{e^{i(y'-y)\sqrt{-t}} - \frac{q-i\sqrt{(-t)}}{q+i\sqrt{(-t)}}\, e^{i(y'+y)\sqrt{(-t)}}\right\} \tag{6.11}$$

Substitution of this value of F in the integral (6.04) yields the Weyl–van der Pol formula for Ψ, which after neglecting small quantities (in the second term) reduces to equation (4.15) characterizing the singularity of Ψ.

Thus, the expression (6.04) for Ψ is established.

It is not difficult to transform the contour integral (6.04) into a series of residues corresponding to the roots of equation (6.05). Let us write this equation in some detail.

Using expression (5.20) for $f_1(y, t)$ and the initial values (5.03) and (5.04)

of the functions f^0 and f^*, we obtain

$$f_1(0, t) = A_1^0(t); \qquad f_1'(0, t) = A_1^*(t) \tag{6.12}$$

and equation (6.05) takes the form

$$A_1^*(t) + qA_1^0(t) = 0 \tag{6.13}$$

Inserting the values (5.18) and (5.19) of the functions A_1^0 and A_1^* we have

$$w_1'(t) - qw_1(t) - \int_0^\infty r(y)\,[f^0(y, t) - qf^*(y, t)]\,w_1(t-y)\,\mathrm{d}y = 0 \tag{6.14}$$

This equation will be referred to as the characteristic equation.

It is essential for the applications that the left-hand side of the characteristic equation is an integral transcendental function of t and that it contains only the functions $f^0(y, t)$ and $f^*(y, t)$, which can be obtained for all values of t by means of numerical integration of the differential equation (5.02) with the initial conditions (5.03) and (5.04). In the case that, from a certain $y = y_1$ upwards, the function $r(y)$ is zero, the integration in (6.14) can be performed and the characteristic equation takes the form

$$\begin{aligned} &w_1'(t-y)\,[f^0(y, t) - qf^*(y, t)] + \\ &\quad + w_1(t-y)\frac{\partial}{\partial y}[f^0(y, t) - qf^*(y, t)] = 0 \qquad (\text{for } y = y_1) \end{aligned} \tag{6.15}$$

We know that in the case of a homogeneous atmosphere the characteristic equation is

$$w_1'(t) - qw_1(t) = 0 \tag{6.16}$$

This equation is obtained from the previous formulae either by putting $r(y) = 0$ in (6.14) or by putting $y = y_1 = 0$ in (6.15). We denote the roots of the characteristic equation by

$$t_1(q), \quad t_2(q), \ldots \tag{6.17}$$

These roots will be functions of the parameter q.

Now let us calculate the residues of the integral (6.04). From equations (6.02) and (6.05) it follows that for $y = 0$ and $t = t_s$ we have

$$\frac{f_2'(0, t) + qf_2(0, t)}{D_{12}(t)} = \frac{1}{f_1(0, t)} \tag{6.18}$$

Further, the derivative with respect to t of the denominator in (6.03)

can be written as

$$\frac{\partial^2 f_1}{\partial y\,\partial t}+q\frac{\partial f_1}{\partial t}=f_1\frac{\partial}{\partial t}\left(\frac{1}{f_1}\cdot\frac{\partial f_1}{\partial y}\right)=-f_1(0,t)\frac{\mathrm{d}q}{\mathrm{d}t} \tag{6.19}$$

Therefore, the residue of F at the pole $t=t_s$ will be equal to

$$\frac{\mathrm{d}t_s}{\mathrm{d}q}\cdot\frac{f_1(y',t_s)}{f_1(0,t_s)}\cdot\frac{f_1(y,t_s)}{f_1(0,t_s)} \tag{6.20}$$

Taking the sum of expressions (6.20) multiplied by e^{ixt_s}, we obtain the desired series expansion of the function Ψ

$$\Psi=\sum_{s=1}^{\infty}e^{ixt_s}\frac{\mathrm{d}t_s}{\mathrm{d}q}\cdot\frac{f_1(y',t_s)}{f_1(0,t_s)}\cdot\frac{f_1(y,t_s)}{f_1(0,t_s)} \tag{6.21}$$

The quantities

$$\frac{f_1(y,t_s)}{f_1(0,t_s)}=f^0(y,t_s)-qf^*(y,t_s) \tag{6.22}$$

can be called the height factors. Let us observe that the height factors are expressible, according to (6.22), in terms of the functions f^0 and f^* which can be calculated directly by means of numerical integration of (5.02).

In the case when q is very large or equal to infinity (horizontal polarization, perfect conductor) the formula (6.21) is to be transformed by means of the identity

$$\frac{q^2f_1^2(0,t_s)}{f_1'^2(0,t_s)}=1 \tag{6.23}$$

The result can be written in the form

$$\Psi=\sum_{s=1}^{\infty}e^{ixt_s}\left(q^2\frac{\mathrm{d}t_s}{\mathrm{d}q}\right)\cdot\frac{f_1(y',t_s)f_1(y,t_s)}{f_1'(0,t_s)f_1'(0,t_s)} \tag{6.24}$$

The quantity

$$q^2\frac{\mathrm{d}t_s}{\mathrm{d}q}=\frac{1}{\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{f_1(0,t)}{f_1'(0,t)}\right)} \tag{6.25}$$

will be finite for $q\to\infty$ Let us note that from (6.19) and (5.29), there follows

$$f_1^2(0,t)\frac{\mathrm{d}q}{\mathrm{d}t}=\int_0^{\infty e^{i\alpha}}f_1^2(y,t)\,\mathrm{d}y \qquad (\alpha=\pi/3) \tag{6.26}$$

Therefore, the series (6.21) can be written as

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \frac{f_1(y', t_s) f_1(y, t_s)}{\int_0^{\infty e^{i\alpha}} f_1^2(y, t_s)\, dy} \tag{6.27}$$

In such a form, it recalls expansions in terms of eigenfunctions. In (6.27) the "eigenvalues" are, however, complex and the "normalizing integrals" in the denominator converge only for complex paths of integration.

In order to express in terms of Ψ the attenuation factor V, it is sufficient to recall the relation (4.18).

$$V(x, y, y', q) = 2\sqrt{(\pi x)} \cdot e^{i\frac{\pi}{4}} \Psi \tag{6.28}$$

In the case of a homogeneous atmosphere, (when $r(y)=0$ and $f_1(y, t)=w_1(t-y)$) the expressions for V resulting from (6.28) and our formulae for Ψ reduce to those which were derived by another method in Chapter 12.

7. *Application of the General Theory to the Superrefraction Case (Schematic Example)*

The analogy with the non-stationary problem of quantum mechanics considered in Section 3, permits us to form a qualitative picture of the superrefraction phenomenon and of the conditions necessary for this phenomenon to occur. The general expression for the attenuation factor, obtained in Section 6, is also suitable for quantitative computations which, however, require rather complicated calculations.

Let us write the expression for the function Ψ connected with the attenuation factor. Putting for brevity,

$$f(y, t) = f^0(y, t) - q f^*(y, t) \tag{7.01}$$

we have from (6.21)

$$\Psi = \sum_{s=1}^{\infty} e^{ixt_s} \frac{dt_s}{dq} f(y', t_s) f(y, t_s) \tag{7.02}$$

where the quantities t_s are the roots of the characteristic equation (6.14). If $r(y)=0$ for $y > y_1$, this equation can be written according to (6.15) in the form

$$w_1'(t-y_1) f(y_1, t) + w_1(t-y_1) f'(y_1, t) = 0 \tag{7.03}$$

where w_1' means the derivative with respect to the argument $(t-y)$ but not the derivative with respect to y. The parameter q enters into this equation through the expression (7.01).

The determination of the conditions for which long-range propagation is possible reduces to the study of the roots of the characteristic equations (6.14) or (7.03). If there is no superrefraction, the imaginary part of the roots of this equation, which according to (7.02) produces attenuation of the waves with increasing distance, will be of the same order as the real part. When superrefraction is present, there exists one or more roots with an abnormally small imaginary part.

In order to have an idea of the conditions under which long-range propagation can occur, let us consider the following schematic example.

We suppose that the function $r(y)$ has the following form:

$$\begin{aligned} r(y) &= (1+\mu^3)(y_1-y) \qquad &(\text{for } 0 < y < y_1) \\ r(y) &= 0 \qquad &(\text{for } y_1 < y) \end{aligned} \tag{7.04}$$

This corresponds to the assumption that the graph of the modified refractive index is a broken line as shown in Fig. 2.

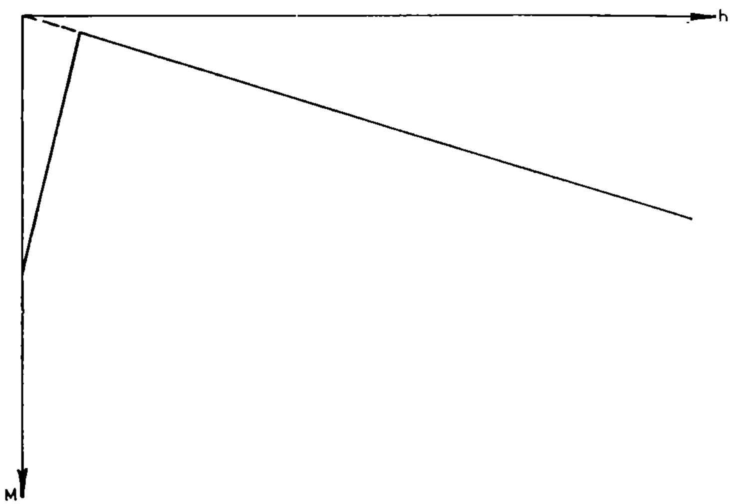

FIG. 2.

If the dielectric constant ε is supposed to vary according to the formula

$$\begin{aligned} \varepsilon &= 1-g(h-h_1) \qquad &(\text{for } h < h_1) \\ \varepsilon &= 1 \qquad &(\text{for } h > h_1) \end{aligned} \tag{7.05}$$

then the parameters μ^3 and y_1 will be equal to

$$\mu^3 = \frac{ag}{2} - 1 \qquad y_1 = h_1 \sqrt[3]{\left(\frac{2k^2}{a}\right)} \tag{7.06}$$

Thus the parameter μ does not depend on the wave length, but the parameter y_1 (the reduced height of the break point) will be proportional to $\lambda^{-2/3}$.

Inserting (7.04) in the equations for f we have

$$\left.\begin{aligned} &\frac{d^2f}{dy^2} + [(1+\mu^3)y_1 - \mu^3 y - t]f = 0 \qquad &(\text{for } y < y_1) \\ &\frac{d^2f}{dy^2} + (y-t)f = 0 \qquad &(\text{for } y > y_1) \end{aligned}\right\} \tag{7.07}$$

Let us introduce instead of t the new parameter

$$\xi_0 = \frac{t-(1+\mu^3)y_1}{\mu^2} \tag{7.08}$$

and instead of y, a new variable

$$\xi = \xi_0 + \mu y \tag{7.09}$$

To the value $y = y_1$ will correspond the value $\xi = \xi_1$, where

$$\mu^2 \xi_1 = t - y_1 \tag{7.10}$$

Equation (7.05) becomes

$$\frac{d^2f}{d\xi^2} = \xi f \qquad (\xi_0 < \xi < \xi_1) \tag{7.11}$$

Its independent solutions are the Airy functions $u(\xi)$ and $v(\xi)$. The functions f^0 and f^* will be equal to

$$\left.\begin{aligned} f^0(y) &= u'(\xi_0)v(\xi) - v'(\xi_0)u(\xi) \\ f^*(y) &= \frac{1}{\mu}[v(\xi_0)u(\xi) - u(\xi_0)v(\xi)] \end{aligned}\right\} \tag{7.12}$$

By virtue of the relation

$$u'(\xi)v(\xi) - v'(\xi)u(\xi) = 1 \tag{7.13}$$

the functions f^0 and f^* will satisfy the initial conditions (5.03) and (5.04). The characteristic equation is obtained if we introduce according to (7.01) the function

$$f(y) = -\frac{1}{\mu}[qv(\xi_0) + \mu v'(\xi_0)]u(\xi) + \frac{1}{\mu}[qu(\xi_0) + \mu u'(\xi_0)]v(\xi) \tag{7.14}$$

and substitute the values of $f(y)$ and $f'(y)$ for $y=y_1$ into the formula (7.03). This characteristic equation can be written in the form

$$\frac{\mu v'(\xi_0)+qv(\xi_0)}{\mu u'(\xi_0)+qu(\xi_0)} = \frac{\mu v'(\xi_1)w_1(\mu^2\xi_1)+v(\xi_1)w_1'(\mu^2\xi_1)}{\mu u'(\xi_1)w_1(\mu^2\xi_1)+u(\xi_1)w_1'(\mu^2\xi_1)} \tag{7.15}$$

Let us suppose that the quantities y_1 and μ are sufficiently large. This means that the "potential well" of Fig. 2 is sufficiently wide and deep. In such a case, the quantities ξ_1 and $\mu^2\xi_1$ (the arguments of the Airy functions on the right-hand side of (7.15)) will be large. By virtue of the asymptotic formulae

$$u(\xi) = \xi^{-\frac{1}{4}} e^{\frac{2}{3}\xi^{3/2}}; \qquad v(\xi) = \frac{1}{2}\xi^{-\frac{1}{4}} e^{-\frac{2}{3}\xi^{3/2}} \tag{7.16}$$

the right-hand side of equation (7.15) will be very small and the characteristic equation reduces, approximately, to

$$\mu v'(\xi_0)+qv(\xi_0) = 0 \tag{7.17}$$

This case will occur when in a sufficiently large interval of the variable h (the height) the gradient of the dielectric constant of air will be negative and larger than $2/a$, where a is the earth's radius; then the curvature of the ray will be larger than that of the earth and the formal expression for the "equivalent radius" becomes negative.

The damping of the wave with increasing horizontal distance is connected with the imaginary part of t and, thus, with the imaginary part of ξ_0; if ξ_0 were real there would be no damping. Attenuation may arise from two causes: absorption by the earth and leakage towards the upper layers of the atmosphere. Absorption by the earth is characterized by the complex parameter q. Equation (7.17) corresponds to the case when the attenuation arises only from absorption by the earth. If we consider the earth as an absolute conductor, we must put $q=0$ for horizontal polarization and $q=\infty$ for vertical polarization.

For $q=0$ equation (7.17) reduces to the form

$$v'(\xi_0) = 0 \qquad (\text{for } q = 0) \tag{7.18}$$

Its roots will be the real negative numbers

$$\xi_0 = -1{\cdot}019; \quad -3{\cdot}248; \quad -4{\cdot}820\ldots \tag{7.19}$$

For $q=\infty$, equation (7.17) becomes

$$v(\xi_0) = 0 \qquad (\text{for } q = \infty) \tag{7.20}$$

and has the roots

$$\xi_0 = -2{\cdot}338; \quad -4{\cdot}088; \quad -5{\cdot}521\ldots \tag{7.21}$$

In these cases, since ξ_0 is real, there is no attenuation.

Equation (7.17) represents an approximation to (7.15) if the quantity ξ_1 (or its real part) is positive and sufficiently large. Since

$$\xi_1 = \xi_0 + \mu y_1$$

this condition will cease to be fulfilled for all roots ξ_0 larger in absolute value than a certain root ξ_0. Therefore, the number of roots with a small imaginary part will be finite.

It is possible to derive a correction to ξ_0, obtainable by taking into account the right-hand side of (7.15) approximately. Let us denote by ξ_{00} a root of equation (7.17) (we will consider this root as the uncorrected value of ξ_0) and by $\Delta\xi_0$ – the correction. This correction is obtained if we insert in the right-hand side of (7.15) the approximate value of ξ_1, namely

$$\xi_1 = \xi_{00} + \mu y_1$$

The correction is given by the formula

$$\Delta\xi_0 = \frac{1}{\sqrt{(-\xi_{00})}}\left(\frac{1}{16}\cdot\frac{\mu^3+1}{\mu^3\xi_1^{3/2}}\cdot e^{-\frac{4}{3}\xi_1^{3/2}} + \frac{i}{4}e^{-2S}\right) \tag{7.22}$$

where

$$S = \frac{2}{3}(\mu^3+1)\xi_1^{3/2} \tag{7.23}$$

Let us observe that the imaginary part of the correction is positive. This corresponds to the fact that the leakage in the upper layers of the atmosphere increases the attenuation.

The condition for the applicability of these approximate formulae is that μy_1 should have a sufficiently large value. Let us recall that according to (7.06) we have

$$\mu y_1 = h_1 \sqrt[3]{[k^2(g-2/a)]} \tag{7.24}$$

where g is the gradient of the dielectric constant taken with the opposite sign, a is the earth's radius, and h_1 is the height of the break point in Fig. 2.

The larger the value of μy_1, the greater the number of almost stationary states (states with small attenuation). It can be said, roughly, that the number of such states equals the number of roots ξ_0, whose absolute value does not exceed the parameter μy_1.

The concept of a ray reflected from the boundary of the layer and from the earth's surface becomes applicable only when the number of almost-stationary states (the number of terms in the series (7.02) with low attenuation) becomes large. Generally, the necessary condition for the applicability of the concepts of geometrical optics is the slow convergence of the series (7.02), (this is just the case when a large number of terms will play a part). If only one or two terms in it are essential (these terms may correspond both to the almost-stationary states and to states with attenuation) then the ray concept is not applicable at all.

8. *Approximate Formulae for Terms With Small Attenuation*

Using a method similar to that which is applied in quantum mechanics, approximate expressions can be derived for the height factors corresponding to terms with small attenuation, and estimates can also be given for that part of the damping coefficient which corresponds to leakage into the upper layers.

We put in (6.01)

$$y+r(y) = p(y) \tag{8.01}$$

and write this equation in the form

$$\frac{d^2f}{dy^2}+[p(y)-t]f = 0 \tag{8.02}$$

In the superrefraction case, the function $p(y)$, which is proportional to the modified refractive index, will have a minimum and will thus increase on both sides of it; to the left of the minimum the largest value of $p(y)$ is $p(0)$ and to the right $p(y)$ increases as the variable y. If t lies between the least value of $p(y)$ and $p(0)$, then the coefficient of f in equation (8.02) vanishes for two values of y which we will denote by y_1 and y_2. In the interval $y_1<y<y_2$ the quantity $p(y)-t$ will be negative, and outside this interval, positive.

In the interval $y_1<y<y_2$ the solution of equation (8.02) can be approximately expressed in terms of Airy functions. We put

$$\int_{y_1}^{y} \sqrt{[t-p(y)]}\,dy = \frac{2}{3}\xi_1^{3/2} \tag{8.03}$$

and

$$\int_{y}^{y_2} \sqrt{[t-p(y)]}\,dy = \frac{2}{3}\xi_2^{3/2} \tag{8.04}$$

and denote by S the sum of these quantities which is independent of y

$$S = \int_{y_1}^{y_2} \sqrt{[t-p(y)]}\,\mathrm{d}y \tag{8.05}$$

We assume the quantity S to be large. With such notation we will have approximately

$$f = \sqrt[4]{\left(\frac{\xi_1}{t-p(y)}\right)} \cdot [A_1 u(\xi_1) + B_1 v(\xi_1)] \tag{8.06}$$

and also

$$f = \sqrt[4]{\left(\frac{\xi_2}{t-p(y)}\right)} [A_2 u(\xi_2) + B_2 v(\xi_2)] \tag{8.07}$$

where

$$\frac{2}{3}\xi_1^{3/2} + \frac{2}{3}\xi_2^{3/2} = S \tag{8.08}$$

and the constants A_1, B_1, A_2, B_2 are connected by the relation

$$A_2 = \frac{1}{2} B_1 \mathrm{e}^{-S}; \qquad B_2 = 2A_1 \mathrm{e}^{S} \tag{8.09}$$

which follow from comparison of the asymptotic expressions for (8.06) and for (8.07) at large values of ξ_1 and ξ_2.

For $y > y_2$ we can define the quantity ξ_2 by means of the equation

$$\int_{y_2}^{y} \sqrt{(p(y)-t)}\,\mathrm{d}y = \frac{2}{3}(-\xi_2)^{3/2} \tag{8.10}$$

and use the previous expression (8.07) for f.

Similarly, for $y < y_2$ we take as a definition of ξ_2 instead of (8.03) the equation

$$\int_{y}^{y_1} \sqrt{(p(y)-t)}\,\mathrm{d}y = \frac{2}{3}(-\xi_1)^{3/2} \tag{8.11}$$

and use the expression (8.06) for f.

Let us choose the constants A, B so that the function f is proportional to $f_1(y, t)$. We must put

$$A_2 = C_1; \qquad B_2 = iC_1 \tag{8.12}$$

and consequently

$$A_1 = \frac{i}{2} C_1 \mathrm{e}^{-S}; \qquad B_1 = 2C_1 \mathrm{e}^{S} \tag{8.13}$$

Then the formulae (8.06) and (8.07) take the form

$$f_1(y, t) = 2C_1 e^S \sqrt[4]{\left(\frac{\xi_1}{t-p(y)}\right)} \cdot \left[v(\xi_1)+\frac{i}{4} e^{-2S} u(\xi_1)\right] \tag{8.14}$$

and

$$f_1(y, t) = C_1 \sqrt[4]{\left(\frac{\xi_2}{t-p(y)}\right)} w_1(\xi_2) \tag{8.15}$$

Similarly, the following approximate expressions are obtained for $f_2(y, t)$:

$$f_2(y, t) = 2C_2 e^S \sqrt[4]{\left(\frac{\xi_1}{t-p(y)}\right)} \left\{v(\xi_1)-\frac{i}{4} e^{2S} u(\xi_1)\right\} \tag{8.16}$$

and

$$f_2(y, t) = C_2 \sqrt[4]{\left(\frac{\xi_2}{t-p(y)}\right)} w_2(\xi_2) \tag{8.17}$$

In this approximation, the Wronskian $D_{12}(t)$ is equal to

$$D_{12}(t) = -2iC_1C_2 \tag{8.18}$$

For $y<y_1$ the functions $v(\xi_1)$ and $u(\xi_1)$ will be both of the same order of magnitude. As a consequence of the smallness of the factor e^{-2S} the second terms in (8.14) and (8.16) will represent small corrections (they may be even less than the error in the whole expression (8.14) or (8.16)). Therefore, in the region $y<y_1$ the functions f_1 and f_2 will be almost proportional to each other.

If the small corrections are neglected, the equation defining t can be written in the form

$$\left(\frac{d\xi}{dy}\right)_0 v'(\xi_0)+qv(\xi_0) = 0 \tag{8.19}$$

Here ξ_0 and $(d\xi/dy)_0$ denote the values of ξ_1 and $d\xi_1/dy$ at $y=0$.

This equation is similar to (7.17). It yields that part of the damping coefficient of the wave which is due to absorption by the earth. Since the complex parameter q which characterized the properties of the ground is known only very roughly, it is sufficient to take the coefficient $(d\xi/dy)_0$ at a rough approximation and to put, according to (7.06) and (7.09),

$$\left(\frac{d\xi}{dy}\right)_0 = \mu = \sqrt[3]{\left(\frac{ag}{2}-1\right)} \tag{8.20}$$

where a is the radius of the earth and g is the gradient of the dielectric

constant taken with the opposite sign. Then (8.19) will reduce to equation (7.17) which was studied in the preceding section. The roots ξ_0 of equation (8.19) will be related to the corresponding values of the parameter t by the relation

$$k\int_0^{h_1^*}\sqrt{\left(-\frac{t}{m^2}+\varepsilon-1+\frac{2h}{a}\right)}\,\mathrm{d}h = \frac{2}{3}(-\xi_0)^{3/2} \tag{8.21}$$

where h_1^* is the lesser of the two values h_1^* and h_2^* of the height h, for which the radical becomes zero.

If ξ_0 is real, then h_1^* and t are real; if ξ_0 is complex, then the evaluation of the integral (8.21) requires an analytic continuation of the interpolation formula for ε into the complex h-plane.

A necessary condition for the applicability of the previous formulae is the smallness of the quantity e^{-2S}, where S has the value (8.05). In conventional units, the integral which expresses S, assumes the form

$$S = k\int_{h_1^*}^{h_2^*}\sqrt{\left[\frac{t}{m^2}-\left(\varepsilon-1+\frac{2h}{a}\right)\right]}\,\mathrm{d}h \tag{8.22}$$

Having determined t from (8.21) one must ascertain that the integral S is sufficiently large for this t.

In the case of a perfect conductor ($q=0$ and $q=\infty$) the approximate values of ξ_0 and t which are obtained from (8.19) and (8.21) will be real. In this case, an approximate expression for the imaginary part of the correction to ξ_0 can be given.

Putting

$$\xi_0 = \xi_0'+i\xi_0'' \tag{8.23}$$

we have

$$\sqrt{(-\xi_0')}\cdot\xi_0'' = \frac{1}{4}\mathrm{e}^{-2S} \tag{8.24}$$

We will not dwell on the derivation of this formula.

Since ξ_0'' is a small quantity, then to the increment $\Delta\xi_0=i\xi_0''$ will correspond the increment $\Delta t=it''=(\mathrm{d}t/\mathrm{d}\xi_0)\cdot\Delta\xi_0$. But the quantity (8.24) multiplied by i is the increment to the integral (8.21). Consequently, we can determine t'' (the imaginary part of t) from the equation

$$t''\frac{\partial}{\partial t}\left[k\int_0^{h_1^*}\sqrt{\left(-\frac{t}{m^2}+\varepsilon-1+\frac{2h}{a}\right)}\,\mathrm{d}h\right] = -\frac{1}{4}\mathrm{e}^{-2S} \tag{8.25}$$

Since the derivative of the integral is negative, the quantity t'' is positive, as it ought to be (positive sign of t'' corresponds to attenuation).

Our formulae permit us to obtain an approximate expression for the quantity $\mathrm{d}q/\mathrm{d}t$. According to (6.19) we have

$$\frac{\mathrm{d}q}{\mathrm{d}t} = -\frac{\partial^2 \log f_1}{\partial y\, \partial t} \tag{8.26}$$

Inserting here the value of f_1 from (8.14) and neglecting small quantities, we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \left(\frac{\mathrm{d}\xi}{\mathrm{d}y}\right)_0 \frac{\partial \xi_0}{\partial t}\left(\frac{v'^2(\xi_0)}{v^2(\xi_0)} - \frac{v''(\xi_0)}{v(\xi_0)}\right) \tag{8.27}$$

Here we can put, in a rough approximation,

$$\left(\frac{\mathrm{d}\xi}{\mathrm{d}y}\right)_0 = \mu; \qquad \frac{\partial \xi_0}{\partial t} = \frac{1}{\mu^2} \tag{8.28}$$

(see formulae (7.08) and (7.09)). Using the differential equation and the boundary conditions for v, we obtain

$$\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{q^2}{\mu^3} - \frac{\xi_0}{\mu} \tag{8.29}$$

The first terms of the series (6.21) for Ψ (those with small attenuation), will be equal, in our approximation, to

$$\sum e^{ixt} \frac{v(\xi_1)v(\xi_1')}{\left(\dfrac{q^2}{\mu^3} - \dfrac{\xi_0}{\mu}\right) v^2(\xi_0)} = \sum e^{ixt} \frac{\mu v(\xi_1) v(\xi_1')}{v'^2(\xi_0) - \xi_0 v^2(\xi_0)} \tag{8.30}$$

where ξ_1' corresponds to the reduced height y'.

If ξ_0 is very large in absolute value, so that asymptotic expressions can be used for $v(\xi_0)$ and $v'(\xi_0)$, then the denominator in this formula will be approximately equal to

$$v'^2(\xi_0) - \xi_0 v^2(\xi_0) = \sqrt{-\xi_0} \tag{8.31}$$

In conclusion, it is necessary to emphasize that the formulae derived in this section are based on rough approximations and are only intended to form a general picture. More exact computations should be based on the rigorous theory proposed in the previous sections.

CHAPTER 15

APPROXIMATE FORMULAE FOR DISTANCE OF THE HORIZON IN THE PRESENCE OF SUPERREFRACTION†

Abstract — A formula is derived for the range of radiowave propagation (horizon distance) in the presence of superrefraction. The formula obtained applies for an atmospheric waveguide next the earth in which the modified refractive index depends on the height according to a hyperbolic law.

1. Introduction

A general formula for the attenuation factor was derived in Chapter 14. The expression obtained is in the form of a contour integral and is applicable for the rather general case of an arbitrary dependence of the refractive index on height. The main difficulty in using our general formula lies in the solution of the differential equation for the height factor. This difficulty can be overcome by using an asymptotic solution of the equation (this method is based on the presence of a large parameter in the equation). When an approximate expression for the height factor has been obtained, the integrand in the contour integral can be written in explicit form and then this integral can be investigated. A qualitative investigation of the integrand allows us to give an estimate of the distance at which the attenuation factor begins to decrease rapidly, in other words, an estimate of the horizon distance.

2. *Initial Formulae*

As was shown in Chapter 14, the field from a vertical and horizontal electric and magnetic dipole is expressed in the general case by means of two Hertz functions, U and W, which satisfy identical differential equations; the boundary conditions for U and W are also of the same type but with different values for the coefficients. Each of the Hertz functions can be

† Fock, 1956.

expressed by means of the attenuation factor V in the following way:

$$U = \frac{e^{iks}}{\sqrt{(sa \sin s/a)}} V \tag{2.01}$$

where a is the radius of the earth, s is the horizontal distance measured along an arc of the earth's globe and $k=2\pi/\lambda$ is the absolute value of the wave vector.

The attenuation factor V is expressed more conveniently in terms of dimensionless quantities: the reduced horizontal distance

$$x = \frac{k}{2m^2} s \tag{2.02}$$

and the reduced heights of the corresponding points (source and observation point):

$$y = \frac{k}{m} h; \qquad y' = \frac{k}{m} h' \tag{2.03}$$

where h and h' are heights in length units and m is the parameter

$$m = \sqrt[3]{\left(\frac{ka}{2}\right)} \tag{2.04}$$

In problems connected with superrefraction the equivalent radius of the earth does not play the role that it plays in the normal refraction case; consequently, we do not introduce it here. In addition to the quantities enumerated, the attenuation factor V depends on the parameter q which enters into the boundary conditions. For the Hertz function U (vertical polarization) the parameter q is equal to

$$q = \frac{im}{\sqrt{(\eta+1)}} \tag{2.05}$$

where η is the complex dielectric constant of the medium. For the Hertz function W (horizontal polarization) the parameter q is equal to

$$q = im\sqrt{(\eta-1)} \tag{2.06}$$

In practice, we can put $q=\infty$ in the last case since both the parameters m and η are large.

Thus, the attenuation factor V is a function of the dimensionless quantities x, y, y', q

$$V = V(x, y, y', q) \tag{2.07}$$

In addition to the attenuation factor V it is convenient to introduce the function Ψ connected with V by the relation

$$V = 2\sqrt{(\pi x)}\, e^{i\frac{\pi}{4}}\, \Psi \tag{2.08}$$

The function Ψ satisfies the differential equation

$$\frac{\partial^2 \Psi}{\partial y^2} + i\frac{\partial \Psi}{\partial x} + [y + r(y)]\Psi = 0 \tag{2.09}$$

where

$$r(y) = m^2(\varepsilon - 1) \tag{2.10}$$

and $\varepsilon = \varepsilon(h)$ is the dielectric constant of the air as a function of height. Equation (2.09) is obtained by a transformation to the dimensionless quantities from the equation

$$\frac{\partial^2 \Psi}{\partial h^2} + 2ik\frac{\partial \Psi}{\partial s} + k^2\left(\frac{2h}{a} + \varepsilon - 1\right)\Psi = 0 \tag{2.11}$$

in which the coefficient of the function Ψ, is proportional to the modified refractive index

$$M(h) = 10^6\left(\frac{\varepsilon - 1}{2} + \frac{h}{a}\right) \tag{2.12}$$

The coefficient of the function Ψ in equation (2.09) is conveniently denoted by a single letter; we put

$$p(y) = y + r(y) \tag{2.13}$$

We have

$$p(y) = m^2\left(\varepsilon - 1 + \frac{2h}{a}\right) \tag{2.14}$$

so that $p(y)$ is, essentially, the same modified refractive index but expressed in terms of the dimensionless height y.

Using the notation of equation (2.13), we can write equation (2.09) in the form

$$\frac{\partial^2 \Psi}{\partial y^2} + i\frac{\partial \Psi}{\partial x} + p(y)\Psi = 0 \tag{2.15}$$

Besides the differential equation (2.15), the function Ψ satisfies the boundary condition

$$\frac{\partial \Psi}{\partial y} + q\Psi = 0 \qquad (\text{for } y = 0) \tag{2.16}$$

At $x=0$, it has a singularity of the form

$$\Psi = \frac{e^{-i\frac{\pi}{4}}}{2\sqrt{(\pi x)}}\left[e^{i\frac{(y-y')^2}{4x}}+e^{i\frac{(y+y')^2}{4x}}\cdot\frac{y+y'+2iqx}{y+y'-2iqx}\right] \tag{2.17}$$

A general expression for the function Ψ in the form of a contour integral was given in Chapter 14. The integrand in the contour integral was expressed in terms of the solutions of the equation

$$\frac{d^2f}{dy^2}+p(y)f = tf \tag{2.18}$$

where t is a complex parameter. (These solutions have been called the height factors in the Introduction.)

In order to form the integrand, it is necessary to know both solutions of equation (2.18); let us denote them by $f_1(y, t)$ and $f_2(y, t)$. For large values of y these functions have the following asymptotic expressions:

$$f_1(y, t) = \frac{c' e^{i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\cdot\exp\left[i\int_\tau^y\sqrt{[p(u)-t]}\,du\right] \tag{2.19}$$

$$f_2(y, t) = \frac{c'' e^{-i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\cdot\exp\left[-i\int_\tau^y\sqrt{[p(u)-t]}\,du\right] \tag{2.20}$$

Here c', c'', τ are constants the values of which are not essential since they drop out from the expression for Ψ. In the case of a homogeneous atmosphere when $p(y)=y$, the functions $f_1(y, t)$ and $f_2(y, t)$ reduce to the complex Airy functions $w_1(t-y)$ and $w_2(t-y)$ and in equations (2.19) and (2.20) we can then put $c'=c''=1$ and $\tau=t$.

Let us put

$$D_{12}(t) = f_1\frac{\partial f_2}{\partial y}-f_2\frac{\partial f_1}{\partial y} \tag{2.21}$$

By virtue of equation (2.18), which is satisfied by f_1 and f_2, this quantity will be independent of y.

Let us denote the values of $\frac{\partial f_1}{\partial y}$ and of $\frac{\partial f_2}{\partial y}$ for $y=0$ by $f_1'(0, t)$ and $f_2'(0, t)$. We form the function

$$F(t, y, y', q) = \frac{1}{D_{12}(t)}f_1(y', t)\left[f_2(y, t)-\frac{f_2'(0, t)+qf_2(0, t)}{f_1'(0, t)+qf_1(0, t)}f_1(y, t)\right] \tag{2.22}$$

The function Ψ is determined for $y' > y$ by the contour integral

$$\Psi = \frac{1}{2\pi i}\int e^{ixt} F(t, y, y', q)\, dt \tag{2.23}$$

taken in a positive direction over a contour enclosing all the poles of the integrand. As shown in Chapter 14, this function satisfies all the conditions set above and yields a solution to our problem.

3. *Normal Refraction Case*

The case of normal refraction is characterized by the modified refractive index $M(h)$ being a monotonously increasing function of the height h; consequently the coefficient $p(y)$ is a monotonously increasing function of y. In this case $f_1(y, t)$ and $f_2(y, t)$ can be expressed approximately by the complex Airy functions of argument ξ defined by the equalities

$$\int_b^y \sqrt{[p(u)-t]}\, du = \frac{2}{3}(-\xi)^{3/2} \tag{3.01}$$

$$\int_y^b \sqrt{[t-p(u)]}\, du = \frac{2}{3}\xi^{3/2} \tag{3.02}$$

where b is a root of the equation

$$p(b) = t \tag{3.03}$$

In the vicinity of $y=b$ the variable ξ will be a holomorphic function of y, namely

$$\xi = \sqrt[3]{[p'(b)]}\,(b-y)+\ldots \tag{3.04}$$

We can put approximately

$$f_1 = \sqrt{\left(-\frac{dy}{d\xi}\right)}\, w_1(\xi), \qquad f_2 = \sqrt{\left(-\frac{dy}{d\xi}\right)}\, w_2(\xi) \tag{3.05}$$

and to the same approximation

$$\frac{\partial f_1}{\partial y} = -\sqrt{\left(-\frac{d\xi}{dy}\right)}\, w'(\xi), \qquad \frac{\partial f_2}{\partial y} = -\sqrt{\left(-\frac{d\xi}{dy}\right)}\, w_2'(\xi) \tag{3.06}$$

from which

$$D_{12} = -2i \tag{3.07}$$

Replacing in these equations y by y' and ξ by ξ', we obtain expressions for $f_1(y,' t)$ and $f_2(y', t)$. The value of ξ corresponding to $y=0$ will be deno-

ted by ξ_0. Using this notation, we obtain for the function F, defined by the formula (2.22) the following approximate expression:

$$F=\frac{i}{2}\sqrt{\left(-\frac{\mathrm{d}y'}{\mathrm{d}\xi'}\right)}\sqrt{\left(-\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)}\cdot w_1(\xi')\left[w_2(\xi)-\frac{w_2'(\xi_0)+q\left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)_0 w_2(\xi_0)}{w_1'(\xi_0)+q\left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)_0 w_1(\xi_0)}\,w_1(\xi)\right] \tag{3.08}$$

When substituted in formula (2.23), this expression can be used for the calculation of the field in both the shadow region and in the illuminated region. In the region of the shadow the attenuation factor (as well as the function Ψ) is calculated by a series of residues corresponding to the roots of the denominator

$$w_1'(\xi_0)+q\left(\frac{\mathrm{d}y}{\mathrm{d}\xi}\right)_0 w_1(\xi_0) = 0 \tag{3.09}$$

In the illuminated region the function Ψ is calculated directly by contour integration, the principal segment of the integration path being near the real negative values of t. But for negative t the quantities ξ_0, ξ and ξ' will also be negative. Supposing these quantities to be sufficiently large, the functions w_1 and w_2 can be replaced by their asymptotic expressions

$$w_1(\xi) = \mathrm{e}^{i\frac{\pi}{4}}\cdot(-\xi)^{-\frac{1}{4}}\,\mathrm{e}^{i\frac{2}{3}(-\xi)^{3/2}} \tag{3.10}$$

$$w_2(\xi) = \mathrm{e}^{-i\frac{\pi}{4}}\cdot(-\xi)^{-\frac{1}{4}}\cdot\mathrm{e}^{-i\frac{2}{3}(-\xi)^{3/2}} \tag{3.11}$$

Such a substitution amounts to the use of the asymptotic expressions (2.19) and (2.20) for $f_1(y, t)$ and $f_2(y, t)$. As a result, the following expression is obtained for the function F (according to formula (2.22)):

$$F=\frac{i}{2}\,\frac{1}{\sqrt[4]{[p(y)-t]}\,\sqrt[4]{[p(y')-t]}}\left\{\exp\left[i\int_y^{y'}\sqrt{[p(u)-t]}\,\mathrm{d}u\right]-\right.$$
$$\left.-\frac{q-i\sqrt{[p(0)-t]}}{q+i\sqrt{[p(0)-t]}}\cdot\exp\left[i\int_0^{y}\sqrt{[p(u)-t]}\,\mathrm{d}u+i\int_0^{y'}\sqrt{[p(u)-t]}\,\mathrm{d}u\right]\right\} \tag{3.12}$$

This formula is a generalization of formula (6.11) of Chapter 14. The latter formula can be obtained from equation (3.12) after substituting zero for $p(y)$.

Inserting (3.12) into the contour integral, we obtain for the attentuation factor an expression, composed of two terms, the first of which corresponds to an incident wave and the second to a wave reflected once (with a Fresnel coefficient) from the earth's surface. The incident wave is a superposition of waves with the phase

$$\omega(t) = xt + \int_{y}^{y'} \sqrt{[p(u)-t]}\,\mathrm{d}u \tag{3.13}$$

and the reflected wave is a superposition of waves with the phase

$$\varphi(t) = xt + \int_{0}^{y} \sqrt{[p(u)-t]}\,\mathrm{d}u + \int_{0}^{y'} \sqrt{[p(u)-t]}\,\mathrm{d}t \tag{3.14}$$

These expressions correspond to geometrical optics. The integrals can be evaluated by the method of stationary phase, where the phase of the incident wave will be equal to the extremum value of $\omega(t)$ and the phase of the reflected wave will be equal to the extremum value of $\varphi(t)$. The function $\omega(t)$ attains its extremum value for t determined from the equation

$$\omega'(t) \equiv x - \frac{1}{2}\int_{y}^{y'} \frac{\mathrm{d}u}{\sqrt{[p(u)-t]}} = 0 \tag{3.15}$$

and the function $\varphi(t)$ for t determined from the equation

$$\varphi'(t) \equiv x - \frac{1}{2}\int_{0}^{y} \frac{\mathrm{d}u}{\sqrt{[p(u)-t]}} - \frac{1}{2}\int_{0}^{y'} \frac{\mathrm{d}u}{\sqrt{[p(u)-t]}} = 0 \tag{3.16}$$

From the viewpoint of geometrical optics the horizon distance is determined by the condition that a reflected wave with a real phase could reach up to this point. The maximum value of t for which this still takes place is $t = p(0)$. This value must at the same time be a root of equation (3.16).

Consequently the following relation must exist between the quantities x, y and y'

$$x = \frac{1}{2}\int_{0}^{y} \frac{\mathrm{d}u}{\sqrt{[p(u)-p(0)]}} + \frac{1}{2}\int_{0}^{y'} \frac{\mathrm{d}u}{\sqrt{[p(u)-p(0)]}} \tag{3.17}$$

This relation gives the formula for the horizon distance in case of normal refraction.

The more exact expression (3.08) for F shows that for the value $t = p(0)$ it is no longer possible to use equation (3.12). Indeed, at this value of t the quantity ξ_0 becomes zero and it is of course inadmissible to use the formulae (3.10) and (3.11). Nevertheless that value of x, determined from

equation (3.17) can be considered to give the boundary separating the illuminated region where the reflection formula applies from the shadow region where the residue series is applicable. In other words, the value (3.17) is that value of x beyond which the field amplitude begins to decrease rapidly. In this case the term "horizon distance" can be used also in diffraction theory.

4. *Asymptotic Integration of a Differential Equation with Coefficient Having a Minimum*

In the presence of superrefraction, the modified refractive index $M(h)$ will not be a monotonic function of the height, but it will have one or several minima corresponding to the separate waveguide channels. We will consider the case of a single minimum; the corresponding height will be called the inversion height and denoted by h_i.

The coefficient $p(y)$ of the differential equation

$$\frac{d^2f}{dy^2}+p(y)f = tf \tag{4.01}$$

is proportional to $M(h)$ and will also have one minimum for the value $y=y_i$ corresponding to $h=h_i$.

We will suppose $p(y)$ to be an analytic function of y. In the region concerned the equation $p(y)=t$ will have two roots $y=b_1$ and $y=b_2$.

For t real and lying between $p(0)$ and $p(y_i)$ both roots will be real; they can be complex for other values of t.

We need for the functions $f_1(y, t)$ and $f_2(y, t)$ an asymptotic expression which would hold uniformly for all the values of y and t considered, including the value $t=p(y_i)$ for which the roots b_1 and b_2 coincide.

The approximate expressions for f_1 and f_2 in terms of the Airy function used in Section 2 are no longer applicable here. They imply the existence of a substitution like (3.01)–(3.02) which defines the variable ξ as a holomorphic function of y and which transforms equation (4.01) approximately to the equation

$$\frac{d^2\omega}{d\xi^2}-\xi\omega = 0 \tag{4.02}$$

where the coefficient of the unknown function has the same monotonic character as in the initial equation. In our case, we must take, as a standard equation, the equation

$$\frac{d^2g}{d\zeta^2}+\left(\frac{1}{4}\zeta^2+\nu\right)g = 0 \tag{4.03}$$

for the parabolic cylinder function instead of equation (4.02) for the Airy function, since equation (4.03) is the most simple equation in which the coefficient of the unknown function has the same character (with a single minimum) as the coefficient $p(y)$. The substitution connecting ζ and y must be chosen in such a way that the quantity $p(y)-t$ becomes zero simultaneously with the quantity $\frac{1}{4}\zeta^2+\nu$ and so that for large values of these quantities correct asymptotic expressions would be obtained. These requirements are satisfied by the substitution

$$\int_{b_1}^{y} \sqrt{[p(y)-t]}\,dy = \frac{1}{2}\int_{-2i\sqrt{\nu}}^{\zeta} \sqrt{[\zeta^2+4\nu]}\,d\zeta \tag{4.04}$$

if the parameter ν is chosen so that

$$\int_{b_1}^{b_2} \sqrt{[p(y)-t]}\,dy = \frac{1}{2}\int_{-2i\sqrt{\nu}}^{2i\sqrt{\nu}} \sqrt{[\zeta^2+4\nu]}\,d\zeta \tag{4.05}$$

The integral on the right-hand side of (4.05) is equal to

$$\frac{1}{2}\int_{-2i\sqrt{\nu}}^{2i\sqrt{\nu}} \sqrt{[\zeta^2+4\nu]}\,d\zeta = i\pi\nu \tag{4.06}$$

Consequently equation (4.05) can be written in the form

$$i\pi\nu = \int_{b_1}^{b_2} \sqrt{[p(y)-t]}\,dy \tag{4.07}$$

It gives ν as a function of t. This function will be holomorphic near $t = p(y_i)$; we have

$$\nu = \frac{p(y_i)-t}{\sqrt{[2p''(y_i)]}}+\dots \tag{4.08}$$

Putting

$$S = \int_0^y \sqrt{[p(y)-t]}\,dy \tag{4.09}$$

$$S_0 = \frac{1}{2}\int_0^{b_1} \sqrt{[p(y)-t]}\,dy+\frac{1}{2}\int_0^{b_2} \sqrt{[p(y)-t]}\,dy \tag{4.10}$$

we can write the substitution (4.04) in the form

$$S-S_0 = \frac{1}{2}\int_0^{\zeta} \sqrt{[\zeta^2+4\nu]}\,d\zeta \tag{4.11}$$

The expression on the right-hand side is equal to

$$\frac{1}{2}\int_0^{\zeta} \sqrt{[\zeta^2+4\nu]}\,\mathrm{d}\zeta = \frac{1}{4}\zeta\sqrt{[\zeta^2+4\nu]}+\nu\log\left(\zeta+\sqrt{[\zeta^2+4\nu]}\right)-\frac{\nu}{2}\log 4\nu \tag{4.12}$$

From this we can conclude that in the case $\zeta > 0$ the quantity $S-S_0+\frac{\nu}{2}\log\nu$ will be a holomorphic function of ν near $\nu=0$ and in the case $\zeta<0$ the quantities $S-S_0-\frac{\nu}{2}\log\nu$ and S will be holomorphic functions of ν. But since at $y=0$ (on the earth's surface), we have $\zeta<0$; we are concerned with the second case in which the sum $S_0+\frac{\nu}{2}\log\nu$ will be also holomorphic near $\nu=0$. This remark will be needed later.

In the asymptotic approximation under consideration, the solutions of equations (4.01) and (4.03) are connected by the relation

$$f = \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}\zeta}\right)}\cdot g \tag{4.13}$$

The solutions of equation (4.03) are functions which can be expressed by means of the parabolic cylinder functions $D_n(z)$ satisfying the equation

$$\frac{\mathrm{d}^2 D_n(z)}{\mathrm{d}z^2}+\left(n+\frac{1}{2}-\frac{1}{4}z^2\right)D_n(z) = 0 \tag{4.14}$$

The functions $D_n(z)$ have been well investigated. We will not enumerate their properties but will refer to the book *Modern Analysis* by Whittaker and Watson where the principal formulae are given. The following series can be taken as a definition of $D_n(z)$:

$$D_n(z) = \frac{2^{-\frac{n}{2}-1}}{\Gamma(-n)}\cdot e^{-\frac{z^2}{4}}\cdot\sum_{m=0}^{\infty}\frac{\Gamma\left(\frac{m-n}{2}\right)}{\Gamma(m+1)}\cdot 2^{\frac{m}{2}}(-z)^m \tag{4.15}$$

Equation (4.03) is obtained from equation (4.14) by replacing z by $\zeta e^{-i\frac{\pi}{4}}$ and $n+1/2$ by $i\nu$. Solutions of equation (4.03) are the functions

$$g_1(\zeta) = D_{i\nu-\frac{1}{2}}\left(\zeta e^{-i\frac{\pi}{4}}\right) \tag{4.16}$$

$$g_2(\zeta) = D_{-i\nu-\frac{1}{2}}\left(\zeta e^{i\frac{\pi}{4}}\right) \tag{4.17}$$

For real ν and ζ, the quantities $g_1(\zeta)$ and $g_2(\zeta)$ will be complex conjugates.

It follows from the properties of $D_n(z)$ that

$$g_1(-\zeta) = e^{-\nu\pi - i\frac{\pi}{2}} g_1(\zeta) + \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2} - i\nu\right)} e^{-\frac{\nu\pi}{4} + i\frac{\pi}{4}} g_2(\zeta) \tag{4.18}$$

and

$$g_2(-\zeta) = e^{-\nu\pi + i\frac{\pi}{2}} g_2(\zeta) + \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2} + i\nu\right)} e^{-\frac{\nu\pi}{4} - i\frac{\pi}{4}} g_1(\zeta) \tag{4.19}$$

In the following we will need asymptotic expressions for $g_1(\zeta)$ and $g_2(\zeta)$. In the region adjoining the positive real axis we have

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} + i\frac{\pi}{8}} e^{i\frac{\zeta^2}{4}} \zeta^{i\nu - \frac{1}{2}} \left(1 + \frac{i\nu^2 - 2\nu - i\frac{3}{4}}{2\zeta^2} + \ldots\right) \tag{4.20}$$

Using equation (4.12) we can also write

$$g_1(\zeta) = e^{\frac{\pi\nu}{4} + i\frac{\pi}{8}} e^{-i\frac{\nu}{2} + i\frac{\nu}{2}\log\nu} \frac{1}{\sqrt[4]{(\zeta^2 + 4\nu))}} \exp\left[\frac{i}{2}\int_0^\zeta \sqrt{(\zeta^2 + 4\nu)}\, d\zeta\right] \tag{4.21}$$

The latter expression holds also for large values of ν. The asymptotic expression for $g_2(\zeta)$ is obtained from (4.21) by replacing i by $-i$.

In order to obtain a formula valid near the negative real axis, we must use relation (4.18). We have

$$g_1(\zeta) = e^{-\frac{3\nu\pi}{4} - i\frac{3\pi}{8}} e^{-i\frac{\nu}{2} + \frac{i\nu}{2}\log\nu} \frac{1}{\sqrt[4]{(\zeta^2 + 4\nu)}} \exp\left[-\frac{i}{2}\int_0^\zeta \sqrt{(\zeta^2 + 4\nu)}\, d\nu\right] +$$
$$+ \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2} - i\nu\right)} e^{-\frac{\nu\pi}{4} + i\frac{\pi}{8}} e^{i\frac{\nu}{2} - \frac{i\nu}{2}\log\nu} \frac{1}{\sqrt[4]{(\zeta^2 + 4\nu)}} \exp\left[\frac{i}{2}\int_0^\zeta \sqrt{(\zeta^2 + 4\nu)}\, d\nu\right] \tag{4.22}$$

We are now in a position to construct the solution of equation (4.01) which satisfies all the requirements set up.

Let us put

$$c_1(\nu) = e^{i\frac{\pi}{8} - \frac{\pi\nu}{4}} e^{i\left(\frac{\nu}{2} - \frac{\nu}{2}\log\nu - S_0\right)} \tag{4.23}$$

By virtue of the property of the quantity S stated above, the exponent in equation (4.23) is a holomorphic function of ν near $\nu = 0$.

An appropriate solution of the equation for the height factor will be the function

$$f_1(y, t) = c_1(\nu)\sqrt{\left(2\frac{dy}{d\zeta}\right)}g_1(\zeta) \tag{4.24}$$

Above the inversion layer (for $S-S_0 \gg 1$) this function has the asymptotic expression

$$f_1(y, t) = \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{\{p(y)-t\}}}\, e^{iS-2iS_0} \tag{4.25}$$

which results from equation (4.21).

Below the inversion layer (for $S_0-S \gg 1$) the asymptotic expression for $f_1(y, t)$ will be

$$f_1(y, t) = \chi_1(\nu)\frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{\{p(y)-t)\}}}\, e^{iS-2iS_0} + e^{-\nu\pi}\frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{\{p(y)-t\}}}\, e^{-iS} \tag{4.26}$$

where we have put

$$\chi_1(\nu) = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2}-i\nu\right)}\, e^{-\frac{\nu\pi}{2}}\, e^{i(\nu-\nu\log\nu)} \tag{4.27}$$

Using the well-known asymptotic expression for the function $\Gamma\left(\frac{1}{2}-i\nu\right)$ it is easy to show that for large positive values of ν the function $\chi_1(\nu)$ tends to unity. Since for $\nu \gg 1$ the second term in equation (4.26) becomes small in comparison with the first term, both expressions for $f_1(y, t)$ will then coincide. It is essential, however, that our expressions for $f_1(y, t)$ are valid not only for large, but also for small values of ν down to $\nu=0$ and that they are holomorphic functions of ν near $\nu=0$.

The appropriate expressions for $f_2(y, t)$ are obtained from the preceding by replacing i by $-i$. In order to write them explicitly, we put

$$c_2(\nu) = e^{-i\frac{\pi}{8}-\frac{\pi\nu}{4}}\, e^{-i\left(\frac{\nu}{2}-\frac{\nu}{2}\log\nu-S_0\right)} \tag{4.28}$$

$$\chi_2(\nu) = \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2}+i\nu\right)}\, e^{-\frac{\nu\pi}{2}}\, e^{-i(\nu-\nu\log\nu)} \tag{4.29}$$

Then it will be

$$f_2(y, t) = c_2(\nu)\sqrt{\left(2\frac{dy}{d\zeta}\right)}\cdot g_2(\zeta) \tag{4.30}$$

and the asymptotic expressions for $f_2(y, t)$ will have the following form: for $S-S_0 \gg 1$

$$f_2(y, t) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{(p(y)-t))}} e^{-iS+2iS_0} \tag{4.31}$$

for $S_0-S \gg 1$

$$f_2(y, t) = \chi_2(\nu) \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}} e^{-iS+2iS_0} + e^{-\pi} \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}} e^{iS} \tag{4.32}$$

Thus the problem of asymptotic integration of the equation for the height factor is solved.

5. *Investigation of the Attenuation Factor*

Now we must insert the expressions found for $f_1(y, t)$ and $f_2(y, t)$ into the formula (2.22) for F and investigate the attenuation factor V or the function Ψ connected with it. For the sake of simplicity, we will limit ourselves to the case $q = \infty$, which corresponds to horizontal polarization. In this case the function F becomes

$$F(t, y, y', \infty) = \frac{1}{D_{12}(t)} f_1(y', t) \left\{ f_2(y, t) - \frac{f_2(0, t)}{f_1(0, t)} f_1(y, t) \right\} \tag{5.01}$$

For the functions (4.24) and (4.30) the Wronskian D_{12} has the constant value

$$D_{12} = -2i \tag{5.02}$$

which is easily derived from the asymptotic expressions (4.25) and (4.31). We will suppose that $y' > y_t$ so that $S(y') - S_0 \gg 1$ and we will consider the two cases when the second height is above the inversion layer and when it is below it. In the first case, we will assume $S(y) - S_0 \gg 1$ which permits expressions (4.25) and (4.31) for f_1 and f_2 to be used. In the second case, we will assume $S_0 - S(y) \gg 1$ and use the expressions (4.26) and (4.32).

In the first case, we have

$$F = \frac{i}{2} \frac{e^{iS(y')-2iS_0}}{\sqrt[4]{[p(y')-t]}\sqrt[4]{[p(y)-t]}} \left[e^{-iS+2iS_0} + \frac{e^{-\pi\nu} - i\chi_2 e^{2iS_0}}{e^{-\pi\nu} + i\chi_1 e^{-2iS_0}} e^{iS-2iS_0} \right] \tag{5.03}$$

The separate terms of this expression admit of an interpretation on the basis of geometrical optics. It is evident that a wave going from above

downward must have the phase factor e^{-iS} and a wave going from below upward must have the phase factor e^{iS}. Expression (5.03) shows that there is only one wave going from above downward, namely, the incident wave with the total phase

$$\omega(t) = xt + S(y') - S(y) \tag{5.04}$$

(we have added here the term xt from the exponential in the integral (2.23)). This phase is the same as the phase of the normal refraction case (equation (3.13)); this is natural, since the incident wave has not yet reached the inversion layer.

As regards the waves going upward from below, there will be an infinite number of them; these waves are obtained by expanding the second term in parentheses in (5.03) in a power series in $e^{-\pi\nu}$. They will correspond to waves reflected several times from the earth's surface and from the inversion layer. The phase of the wave reflected once from the earth's surface will be

$$\varphi(t) = xt + S(y') + S(y) + \operatorname{arc}(\chi_2/\chi_1) \tag{5.05}$$

This expression differs from equation (3.14) in its last term which cannot be obtained from geometrical optics. This term is equal to

$$\operatorname{arc}\frac{\chi_2}{\chi_1} = \operatorname{arc}\frac{\Gamma\left(\dfrac{1}{2} - i\nu\right)}{\Gamma\left(\dfrac{1}{2} + i\nu\right)} + 2\nu\log\nu - 2\nu \tag{5.06}$$

It vanishes for large positive ν but it plays an important part for small ν since, owing to this term, the whole phase $\varphi(t)$ remains a holomorphic function of ν near $\nu = 0$ in other words, near $t = p(y_i)$.

Now let us examine the case when the point y is below the inversion layer, and $S_0 - S \gg 1$.

Using expressions (4.26) and (4.32) and the equality

$$\chi_1(\nu)\chi_2(\nu) - e^{-2\pi\nu} = 1 \tag{5.07}$$

we obtain after some calculations

$$F = \frac{e^{iS(y') - 2iS_0}}{\sqrt[4]{\{p(y') - t\}}\,\sqrt[4]{\{p(y) - t\}}} \cdot \frac{\sin S(y)}{\chi_1 e^{-2iS_0} - ie^{-\pi\nu}} \tag{5.08}$$

In this case, there is not one but an infinite number of waves going downward from above since to the incident wave are added waves reflected from the inversion layer playing the role of an upper boundary. Moreover,

there is an infinite number of waves reflected from the earth and going upward from below. All these waves are obtained formally by expanding equation (5.08) in a geometrical progression in powers of $e^{-\pi\nu}$.

The total phase of the wave which did not undergo a reflection from the earth is equal to

$$\omega(t) = xt+S(y')-S(y)-\operatorname{arc}\chi_1 \tag{5.09}$$

or

$$\omega(t) = xt+S(y')-S(y)+\frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} \tag{5.10}$$

and the total phase of the wave reflected once only is

$$\varphi(t) = xt+S(y')+S(y)+\frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} \tag{5.11}$$

The expression for $\omega(t)$ does not coincide with that given by equations (3.13) or (5.04), which is natural, since the incident wave passed through the inversion layer. Expression (5.11) differs from (5.05) by the factor $\frac{1}{2}$ in the additional term.

Up to now, we have spoken of the phases of the different terms of the integrand. Each such term gives after integration over t a corresponding term in the attenuation factor. If these integrals over t are evaluated by the method of stationary phase, then each one gives a wave with a phase equal to the extremum value of the phase of the integrand.

It is understood that this method of evaluating the attenuation factor can be used only in the illuminated region, while in the region of the shadow, residue series must be used.

6. *Formula for the Horizon Distance*

We defined the horizon distance for normal refraction (Section 2) as that value of the horizontal range x which would give the boundary between the region of applicability of the reflection formula and the region of applicability of the residue series. For this value of x the extremum of the phase of the reflected wave must correspond to the least value of x for which the phase itself is still real.

If superrefraction is present there are many reflected waves. But we can expect that the principal part will be played by the wave reflected only once from the earth's surface. Inasmuch as the "horizon distance" is not a strictly defined concept, we can make it more precise if we mean by it the horizon distance for a wave reflected once only.

The phases of such a wave are found in Section 5. According to equations (5.05) and (5.11), we will have for $y' > y_i$, $y > y_i$

$$\varphi(t) = xt + \int_0^{y'} \sqrt{[p(u)-t]}\, du + \int_0^{y} \sqrt{[p(u)-t]}\, du + \operatorname{arc}\frac{\chi_2}{\chi_1} \tag{6.01}$$

and for $y' > y_i$, $y < y_i$

$$\varphi(t) = xt + \int_0^{y'} \sqrt{[p(u)-t]}\, du + \int_0^{y} \sqrt{[p(u)-t]}\, du + \frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} \tag{6.02}$$

These formulae can be written conveniently by putting

$$S^*(y, t) = \int_0^{y} \sqrt{[p(u)-t]}\, du + \frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} \qquad (y > y_i) \tag{6.03}$$

$$S^*(y, t) = \int_0^{y} \sqrt{[p(u)-t]}\, du \qquad (y < y_i) \tag{6.04}$$

Then, both for $y > y_i$ and for $y < y_i$ we have

$$\varphi(t) = xt + S^*(y', t) + S^*(y, t) \tag{6.05}$$

We observe that S^* is a holomorphic function of t near $t = p(y_i)$.

Reasoning as in Section 3, we obtain the following expression for the horizon distance

$$x = -\left\{\frac{\partial S^*(y', t)}{\partial t} + \frac{\partial S^*(y, t)}{\partial t}\right\}_{t=p(y_i)} \tag{6.06}$$

Let us write this expression in a more explicit form. According to equation (4.08), near $t = p(y_i)$ we have

$$\nu = \frac{p(y_i)-t}{\sqrt{[2p''(y_i)]}} \tag{6.07}$$

On the other hand, near $\nu = 0$ we have

$$\frac{1}{2}\operatorname{arc}\frac{\Gamma\left(\frac{1}{2}-i\nu\right)}{\Gamma\left(\frac{1}{2}+i\nu\right)} = (C + 2\log 2)\nu + \dots \tag{6.08}$$

and consequently

$$\frac{1}{2}\operatorname{arc}\frac{\chi_2}{\chi_1} = \nu(C - 1 + \log 4\nu) + \dots \tag{6.09}$$

where $C = 0{\cdot}577$ is the Euler constant. Therefore for $y > y_i$ we have

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2}\int_0^{y} \frac{du}{\sqrt{[p(u)-t]}} + \frac{1}{\sqrt{[2p''(y_i)]}}(C + \log 4\nu) \tag{6.10}$$

This expression has a limit for $t \to p(y_i)$, $\nu \to 0$. If $y < y_i$ the last term in (6.10) is absent and the value $t = p(y_i)$ can be inserted directly into the integral. Consequently, for $y < y_i$ we have

$$-\frac{\partial S^*}{\partial t} = \frac{1}{2}\int_0^y \frac{\mathrm{d}u}{\sqrt{[p(u) - p(y_i)]}} \tag{6.11}$$

The presence of the second term in the formula (6.02) entails the dependence of the horizon distance on the wavelength. In order to show this dependence, let us return from the reduced coordinates x, y to the usual coordinates s, h, where s is the horizontal range and h is the height. Denoting the modified refractive index without the factor 10^6 by $\mu(h)$, we have

$$p(y) = 2m^2\mu(h) \tag{6.12}$$

where m is the quantity (2.04). Instead of t we introduce the parameter τ connected with t by relation

$$t = 2m^2\tau \tag{6.13}$$

Then

$$\int_0^y \sqrt{[p(u) - t]}\,\mathrm{d}u = k\int_0^h \sqrt{[2\mu(h) - 2\tau]}\,\mathrm{d}h \tag{6.14}$$

$$xt = ks\tau \tag{6.15}$$

The quantity ν will be approximately equal to

$$\nu = \frac{k}{\sqrt{[\mu''(h_i)]}}[\mu(h_i) - \tau] \tag{6.16}$$

The formula for the horizon distance is obtained from the condition

$$\frac{1}{k}\frac{\partial \varphi}{\partial \tau} = 0 \qquad (\text{for } \tau = \mu(h_i)) \tag{6.17}$$

where the phase φ is supposed to be expressed in terms of the new quantities.

We put

$$F(h) = \int_0^h \frac{\mathrm{d}h}{\sqrt{[2\mu(h) - 2\mu(h_i)]}} \qquad (\text{at } h < h_i) \tag{6.18}$$

$$F(h) = \lim_{\tau \to \mu(h_i)} \left\{ \int_0^h \frac{\mathrm{d}h}{\sqrt{[2\mu(h) - 2\tau]}} + \frac{1}{\sqrt{[\mu''(h_i)]}}\left[C + \ln\frac{4k(\mu(h_i) - \tau)}{\sqrt{[\mu''(h_i)]}} \right] \right\}$$
$$(\text{at } h > h_i) \tag{6.19}$$

Then the formula for the horizon distance obtained from condition (6.17) can be written in the form

$$s = F(h')+F(h) \tag{6.20}$$

Let us compare the values of the horizon distance for equal heights but for different wavelengths. The wavelength enters into the expression for $F(h)$ only for $h > h_i$ and only into the logarithmic term. Let the horizon distance equal s_1 for $\lambda = \lambda_1 = 2\pi/k_1$, and s_2 for $\lambda = \lambda_2 = 2\pi/k_2$. Forming the difference of expressions (6.20), we obtain for $h > h_i$

$$s_2 - s_1 = \frac{2}{\sqrt{[\mu''(h_i)]}} \log\frac{k_2}{k_1} = \frac{2}{\sqrt{[\mu''(h_i)]}} \log\frac{\lambda_1}{\lambda_2} \tag{6.21}$$

and for $h < h_i$

$$s_2 - s_1 = \frac{1}{\sqrt{[\mu''(h_i)]}} \log\frac{k_2}{k_1} = \frac{1}{\sqrt{[\mu''(h_i)]}} \log\frac{\lambda_1}{\lambda_2} \tag{6.22}$$

This difference depends (besides the ratio of the wavelengths) only on the behaviour of the modified refractive index near its minimum.

Now we will apply our general formulae to the case when the modified refractive index $\mu(h)$ depends on the height according to a hyperbolic law

$$\mu(h) = \mu(h_i) + \frac{1}{a}\,\frac{(h-h_i)^2}{h+l} \tag{6.23}$$

where a is the radius of the earth's globe; l is a parameter. In this case

$$\mu''(h_i) = \frac{2}{a(h_i+l)} \tag{6.24}$$

The integrals in $\varphi(t)$ will in general be elliptic but in the case $\tau = \mu(h_i)$ they reduce to elementary ones and we obtain for $F(h)$ the following expressions:

$$F(h) = -\sqrt{[2a(h+l)]} + \sqrt{[2al]} + \\ + \sqrt{\left(\frac{a(h_i+l)}{2}\right)} \left\{ \log\frac{\sqrt{(h_i+l)}+\sqrt{(h+l)}}{\sqrt{(h_i+l)}-\sqrt{(h+l)}} - \log\frac{\sqrt{(h_i+l)}+\sqrt{l}}{\sqrt{(h_i+l)}-\sqrt{l}} \right\} \tag{6.25}$$

$$\text{for } h < h_i$$

and

$$F(h) = +\sqrt{[2a(h+l)]} + \sqrt{(2al)} - \\ \sqrt{\left(\frac{a(h_i+l)}{2}\right)} \left\{ \log\frac{\sqrt{(h+l)}+\sqrt{(h_i+l)}}{\sqrt{(h+l)}-\sqrt{(h_i+l)}} + \log\frac{\sqrt{(h_i+l)}+\sqrt{l}}{\sqrt{(h_i+l)}-\sqrt{l}} \right\} + \Delta s \tag{6.26}$$

$$\text{for } h > h_i$$

where

$$\Delta s = \sqrt{\left(\frac{a(h_i+l)}{2}\right)}\left\{C_1+\frac{1}{2}\log\frac{2k^2(h_i+l)^3}{a}\right\} \tag{6.27}$$

Here

$$C_1 = 7\log 2-4+C = 1{\cdot}429 \tag{6.28}$$

For comparison, we remark that, as is well known, the horizon distance in the absence of refraction is equal to

$$s' = \sqrt{(2ah')}+\sqrt{(2ah)} \tag{6.29}$$

Thus, the increase in the horizon distance due to refraction is equal to

$$s-s' = [F(h')-\sqrt{(2ah')}]+[F(h)-\sqrt{(2ah)}] \tag{6.30}$$

In all the preceding reasoning we supposed that the heights h and h' are small in comparison with the radius of the earth a. But the preceding formulae are applicable also in the case of a wave coming from infinity (for example, from the sun). The difference $F(h')-\sqrt{(2ah')}$ has a finite limit for $h \to \infty$

$$\lim_{h'\to\infty}[F(h')-\sqrt{(2ah')}] = \sqrt{(2al)}-\sqrt{\left(\frac{a(h_i+l)}{2}\right)}\log\frac{\sqrt{(h_i+l)}+\sqrt{l}}{\sqrt{(h_i+l)}-\sqrt{l}}+\Delta s \tag{6.31}$$

Replacing the first two terms in equation (6.30) by their limit values, we obtain for the increase in the horizon distance the following expressions

$$s-s' = 2\sqrt{(2al)}-\sqrt{[2a(h+l)]}-\sqrt{(2ah)}+\Delta s+\sqrt{\left(\frac{a(h_i+l)}{2}\right)}\left[\log\frac{\sqrt{(h_i+l)}+\sqrt{(h+l)}}{\sqrt{(h_i+l)}-\sqrt{(h+l)}}-2\log\frac{\sqrt{(h_i+l)}+\sqrt{l}}{\sqrt{(h_i+l)}-\sqrt{l}}\right] \tag{6.32}$$

(for $h < h_i$)

and

$$s-s' = 2\sqrt{(2al)}+\sqrt{[2a(h+l)]}-\sqrt{(2ah)}+2\Delta s-\sqrt{\left(\frac{a(h_i+l)}{2}\right)}\left[\log\frac{\sqrt{(h+l)}+\sqrt{(h_i+l)}}{\sqrt{(h+l)}-\sqrt{(h_i+l)}}+2\log\frac{\sqrt{(h_i+l)}+\sqrt{l}}{\sqrt{(h_i+l)}-\sqrt{l}}\right] \tag{6.33}$$

(for $h > h_i$)

To this increase in the distance corresponds the "lead angle"

$$\delta = \frac{s-s'}{a} \tag{6.34}$$

Since the present theory does not take into account the refraction in the high layers of the atmosphere, it is necessary, for a comparison with the observed lead angle, to add the value of the normal refraction on the horizon to the quantity (6.34).

CHAPTER 16

ON RADIOWAVE PROPAGATION NEAR THE HORIZON IN THE PRESENCE OF SUPERREFRACTION†

Abstract — This paper is devoted to the computation, for several typical examples, of anomalous radiowave propagation near the horizon when an inversion layer (which does not change in the horizontal directions) exists near the earth. Curves are constructed for the attenuation factor in the case when the transmitting antenna is situated high above the inversion layer and the receiving antenna is within the inversion layer at a low elevation (or conversely).

The results obtained indicate the expediency of introducing the notion of different horizons when analysing long-range propagation; our results allow an estimate of the possible values of the attenuation factor at each of the horizons and also show the dependence of the attenuation factor near the respective horizons on distance and on wavelength. This may be of value in analysing the propagation of decimeter, centimeter and shorter wave-lengths in the troposphere.

1. Introduction

The theory of radiowave propagation above a spherical earth in the presence of an inhomogeneous atmosphere for which the refractive index depends only on the height, was worked out in Chapters 14 and 15. In Chapter 15 the attenuation factor in an inhomogeneous atmosphere near the horizon is investigated and the concept of the horizon is defined for an inhomogeneous (stratified) atmosphere of any kind. The definition of the horizon introduced in the case of an inhomogeneous atmosphere without inversion of the modified refractive index is essentially the same as the definition of the boundary of the shadow according to the laws of geometrical optics. If an inversion of the modified refractive index exists, then the horizon has to be found from more careful wave considerations; in this case, its position depends on the wavelength.

If it is assumed that beyond the horizon the attenuation factor decreases rapidly with increasing distance, then the horizon distance can conditionally be considered as determining the range of radiowave propagation.

† Fock, Wainstein and Belkina, 1956.

In this way a simple formula can be obtained for the range of radiowave propagation with superrefraction. This formula involves the heights of the receiving and transmitting antennae, the wavelength and the parameters characterizing the M-profile. The range formula takes an especially simple form if the modified refractive index is supposed to depend on the height according to a hyperbolic law (Section 5, Chapter 15).

The analysis of long-range propagation on the basis of the horizon concept, given in Chapter 15, requires, however, certain improvements. First of all, it is desirable to elucidate the values assumed by the attenuation factor at the horizon and to clarify how the attenuation factor near the horizon depends on the distance, the wavelength and the parameters of the inversion layer (the height of this layer, its average gradient, etc.). In order to do this, it is obviously necessary to calculate the attenuation factor for certain particular cases as the explicit solution of this problem is practically impossible. If we show how rapidly the attenuation factor decreases in the shadow region (beyond the horizon) and how rapidly it increases to a value of the order of unity when moving off from the horizon into the illuminated region, then we can estimate to what extent the horizon determines the range of radiowave propagation in practical cases.

Since the computations of the attenuation factor in cases of superrefraction take much labour and time, they can be made only for a small number of typical examples. It is practically impossible to perform any exhaustive calculations, as is the case for normal radiowave propagation. Accordingly, we have limited ourselves to the calculation of the attenuation factor as a function of the dimensionless coordinate ζ in four cases. This enables us to construct the dependence of the attenuation factor on the horizontal distance between corresponding points, for a fixed M—curve and for fixed heights of these points, for four wavelengths, proportional to 1 : 3 : 9 : 27 (see Section 7).

In this way, it appears possible to make the meaning of the horizon distance and of the range of propagation more precise and also to answer a number of the questions formulated above, in particular, the question of the dependence of the long-range propagation on wavelength.

Let us recall that the analysis of anomalous propagation, given in Chapter 15, is only applicable if one of the corresponding points is situated above the inversion layer, while the other point can lie either within this layer or below it. Consequently, when computing the attenuation factor we have limited ourselves to the case when one point lies high above the inversion layer and the other is within the layer at a height equal to one-fifth the height of the inversion point.

2. *On the Horizon Concept in the Presence of a Tropospheric Waveguide Adjacent to the Earth*

Let us consider in more detail the horizon concept when a waveguide (inversion layer) exists near the earth.

First, let us recall the ray treatment of normal and anomalous propagation. In case of a homogeneous atmosphere the modified refractive index is a linear function of the height.

The rays, issuing from the source Q, have the shape of curves inverted convexly to the s-axis (Fig. 1a) on the (sh)-plane (s is the distance along the earth, h is the height). The horizon OO' is determined by the ray QOO' which touches the earth at the point O. To the right of the horizon line OO' is the shadow region into which the field penetrates only because of diffraction; to the left of this line lies the illuminated region. For observation points in the illuminated region to the left of the OO' horizon, the reflection formula according to which the field is obtained as a result of the interference of the direct ray OP with the ray $QP'P$ reflected from the earth, is approximately applicable.

Rays from the source Q situated within an atmospheric waveguide, adjacent to the earth, of height h_i (Fig. 1b) are convex upwards (*from* the s axis) within the waveguide and are convex downwards (as in Fig. 1a) above the waveguide. Consequently, the ray $Q1$ passes into space above the waveguide but the ray $Q2$ is "trapped" within the waveguide. These two kinds of rays are separated by the limiting ray QO which approaches the height $h=h_i$ asymptotically at $s\to\infty$. Besides the direct rays the space above the inversion layer is traversed by rays reflected from the earth, as $Q1''1'$ for example, which are separated from the trapped rays by another limiting ray $QO''O'$ approaching the height h_i asymptotically after a single reflection from the earth. All rays issuing from a source within the angle OQO'' formed by both the limiting rays are trapped.

In this example, the laws of geometrical optics lead to the conclusion that a horizon is absent both within and above the waveguide. Indeed, observation points situated above the waveguide to the right of the rays 1 and 1′ are reached by direct rays issuing from Q within the angle $1QO$ and by reflected rays issuing from Q within the angle $1''QO''$. The whole space above the waveguide to the right of the rays 1 and 1′ is filled by the rays and, consequently, there is no region of geometrical shadow and, therefore, the horizon is absent.

However, it is easy to see that the laws of geometrical optics are not applicable to the limiting rays QO and $QO''O'$ and to rays close to them.

From the preceding, it is clear that (according to the laws of geometrical optics) precisely these rays would carry the electromagnetic energy to long distances above the waveguide. Hence, it follows that in order to solve

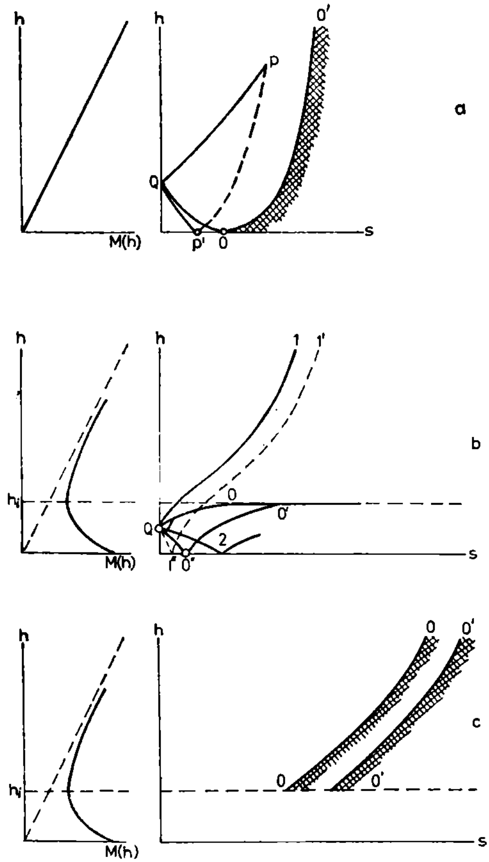

FIG. 1. The concept of the horizon, *a* — for normal refraction; *b* — for superrefraction according to geometrical optics; *c* — for superrefraction according to wave optics.

the question concerning the horizon and the range of propagation in the presence of superrefraction, wave considerations must be used.

This was done in Chapter 15, where it is shown that there is a certain boundary $O'O'$ (Fig. 1c) in the space above the waveguide, to the right of which a ray reflected from the earth cannot penetrate. This boundary

$O'O'$ is the horizon in the presence of an inversion layer since the field can penetrate to the right of this boundary, i.e., into the shadow region, only owing to diffraction (as in the case of Fig. 1a).

Besides the boundary $O'O'$ there is also the boundary OO, to the right of which direct rays which do not undergo reflection from the earth, cannot penetrate. The boundary $O'O'$ lies to the right of the boundary OO since a ray, reflected from the earth, runs to the right of a direct ray and parallel to it (see the rays $Q1$ and $Q1''1'$ on Fig. 1b). Direct rays do not pass into the $OO-O'O'$ band, consequently, the total field in this band is not subject to the ray treatment. In the region to the left of the boundary OO the total electromagnetic field is obtained by superposition of the direct and the reflected rays.

Because of the part played by the boundary OO, which gives the limits of applicability of the reflection formula, it is expedient to introduce a special designation for it: we call it the horizon for the direct wave. In contrast, we call the boundary $O'O'$ the horizon for the reflected wave. Whilst these horizons coincide for normal propagation, they must be differentiated in the case of anomalous propagation.

The horizons $O'O'$ and OO on Fig. 1c replace, in the wave picture, the limiting rays $QO''O'$ and QO (Fig. 1b), obtained from geometrical optics.

These general considerations will be made more precise in Section 4.

3. *Fundamental Formulae*

The attenuation factor V in an inhomogeneous atmosphere for which the refractive index depends only on height can be represented in form of a contour integral

$$V(x, y', y) = e^{-i\frac{\pi}{4}} \sqrt{\left(\frac{x}{\pi}\right)} \int_C e^{ixt} F(t, y', y)\, dt \tag{3.01}$$

When an inversion layer adjacent to the earth is present, and if one of the corresponding points lies above the layer and the other within it, then the following approximate expression (see Chapter 15, formula (5.08)) can be taken for the integrand F

$$F(t, y', y) = \frac{e^{i[S(y')-2S_0]} \sin S(y)}{\sqrt[4]{[p(y')-t]}\,\sqrt[4]{[p(y)-t]}\,[\chi(v)\, e^{-2iS_0} - i\, e^{-\pi v}]} \tag{3.02}$$

Here y' and y are the non-dimensional heights of the source and the observation point (it is supposed that $y' > y_i$ while $y < y_i$ where y_i is the non-dimensional height of the inversion point); x is the non-dimensional horizontal distance between the source and the observation point and $p(y)$ is a func-

tion connected with the modified refractive index $M(h)$ by the formula

$$p(y) = \frac{2m^2}{10^6} M(h) = 2m^2\left(n-1+\frac{h}{a}\right); \qquad m = \left(\frac{ka}{2}\right)^{1/3} \tag{3.03}$$

where n is the refractive index of air; a is the radius of the earth.

We suppose that the function $M(h)$ has the same shape as on Figs. 1b and 1c. Consequently, for a given t, the equation

$$p(y)-t = 0 \tag{3.04}$$

has two roots y_1 and y_2. These roots are real and positive for $p(y_i) < t < p(0)$; they are complex conjugates for $t < p(y_i)$ and they coincide for $t = p(y_i)$ when $y_1 = y_2 = y_i$. In general, there can exist other roots (negative or complex) besides these two but they are of no importance.

The quantities $S(y)$, $S(y')$ and S_0 are given by the formulae

$$S(y) = \int_0^y \sqrt{[p(y)-t]}\,\mathrm{d}y; \qquad S(y') = \int_0^{y'} \sqrt{[p(y)-t]}\,\mathrm{d}y$$
$$S_0 = \frac{1}{2}\int_0^{y_1} \sqrt{[p(y)-t]}\,\mathrm{d}y + \frac{1}{2}\int_0^{y_2} \sqrt{[p(y)-t]}\,\mathrm{d}y \tag{3.05}$$

where for $t < p(y_i)$ the radical $\sqrt{(p(y)-t)}$ must be taken for y real and positive in the arithmetic sense. In order to evaluate S_0 for $t < p(y_i)$ the radical $\sqrt{\{p(y)-t\}}$ must be continued analytically into the region of complex y. We will suppose that $p(y)$ is an analytic function (see (3.18) below) admitting of such continuation.

The quantity ν is defined by the formula

$$\nu = \frac{1}{i\pi}\int_{y_1}^{y_2} \sqrt{[p(y)-t]}\,\mathrm{d}y \tag{3.06}$$

For real values of t the quantity ν is also real while the sign of ν is chosen from the following considerations. For $y \approx y_i$ the function $p(y)$ can be replaced by the first terms of the Taylor series

$$p(y) = p(y_i) + \frac{1}{2}p''(y_i)(y-y_i)^2$$

Then the integral (3.06) can be calculated and we obtain the following approximate formula valid for $t \approx p(y_i)$

$$\nu = \frac{p(y_i)-t}{\sqrt{[2p''(y_i)]}} \tag{3.07}$$

In conformity with this formula, we take $\nu > 0$ for $t < p(y_i)$ and $\nu < 0$ for $t > p(y_i)$. For $p(y_i) < t < p(0)$ formula (3.06) is interpreted as follows:

$$\nu = -\frac{1}{\pi}\int_{y_1}^{y_2} \sqrt{[t-p(y)]}\, dy \tag{3.08}$$

where $\sqrt{[t-p(y)]} > 0$ and $y_1 < y_2$.

The function $\chi(\nu)$ is defined by the formula

$$\chi(\nu) = \frac{\sqrt{(2\pi)}\, e^{-\frac{\pi}{2}\nu + i(\nu - \nu \ln \nu)}}{\Gamma\left(\frac{1}{2} - i\nu\right)} \tag{3.09}$$

where for $\nu > 0\,(t < p(y_i))$ the principal value of ln ν is to be taken†. Then we have

$$\chi(\nu) \to 1 \qquad \text{for } \nu \to \infty \tag{3.10}$$

When evaluating the attenuation factor for large values of y, it is necessary to take into account that for $y \to \infty$ the function $p(y)$ satisfies the following relation:

$$\lim_{y\to\infty} [p(y) - y] = 0 \tag{3.11}$$

Consequently, representing $S(y')$ in the form

$$S(y') = \int_0^{y'} \sqrt{(y-t)}\, dy + \int_0^{y'} \left[\sqrt{[p(y)-t]} - \sqrt{(y-t)}\right] dy$$

we see that for $y \to \infty$ the first component increases without limit (the infinite part equals $2/3 \cdot y'^{3/2} - t\sqrt{y'}$) and the second tends to a finite limit if the difference $p(y) - y$ approaches zero rapidly enough (e.g. like the function $p(y)$ defined by equations (3.18) and (3.19) below).

Let us introduce the quantity ξ_0 as the limit

$$\xi_0 = \lim_{y'\to\infty} \left[S(y') - 2S_0 - \frac{2}{3} y'^{3/2} + t\sqrt{y'}\right] \tag{3.12}$$

Substituting the following expression for $S - 2S_0$ valid for large values of y'

$$S(y') - 2S_0 = \frac{2}{3} y'^{3/2} - t\sqrt{y'} + \xi_0$$

and replacing the quantity $\sqrt[4]{[p(y') - t]}$ in the denominator of (3.02) by $\sqrt[4]{y'}$, we obtain the attenuation factor in the form

$$V(x, y', y) = \sqrt[4]{\left(\frac{x^2}{y'}\right)} \cdot e^{i\frac{2}{3}y'^{3/2}}\, V_1(\zeta, y) \tag{3.13}$$

† In Chapters 16 and 17 the symbol "log" is used to denote common logarithms (to the base 10), while the symbol "ln" denotes natural logarithms (to the base e=2.718...).

where

$$V_1(\zeta, y) = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\pi}} \int_C e^{i\zeta t}\, \Psi(t, y)\, dy \tag{3.14}$$

and

$$\Psi(t, y) = \frac{e^{i\xi_0} \sin S(y)}{\sqrt[4]{[p(y)-t]}\,[\chi(\nu)\, e^{-2iS_0} - ie^{-\pi\nu}]} \tag{3.15}$$

The function $V_1(\zeta, y)$ is related to the attenuation factor V by the same formula (3.13) as in the theory of normal radiowave propagation. Just as in this latter theory, it is natural to call V_1 the attenuation factor of plane waves. Since in the following we shall calculate only V_1, we shall often call V_1 simply the attenuation factor.

The variable ζ which was introduced in equations (3.13) and (3.14) is equal to

$$\zeta = x - \sqrt{y'} \tag{3.16}$$

The geometrical meaning of of the quantity ζ follows from Fig. 2, where T denotes the point at which the incident plane wave (or a spherical wave from a remote source) touches the earth's surface. The quantity ζ

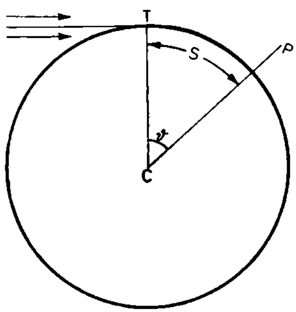

FIG. 2. Geometrical meaning of the quantity ζ.

is connected with the angle $\vartheta = TCP$ (P is the observation point, C is the centre of the earth), or with the corresponding distance $a\vartheta$ measured along the earth, by the relations

$$\zeta = m\vartheta = m\frac{s}{a}\,; \qquad m = \left(\frac{ka}{2}\right)^{1/3} \tag{3.17}$$

Let us note that the point of contact T has a purely geometrical meaning; it corresponds to the path of a fictitious ray in a homogeneous atmosphere.

The infinite contour C in the plane of the complex variable t, over which the integrals for V and V_1 are taken, is arbitrary to a considerable degree and is to be chosen so that the integral can be evaluated with the least difficulty, in particular, so that the principal part of the integration path is as small as possible. Also, the contour C should enclose all the poles of the integrand so that they lie above and to the left of the contour C running in a positive direction. It is most convenient to take the contour shown in Fig. 3, with its break-point either at $t=p(y_i)$ or somewhat to the left (see the end of Section (6)).

As is seen from (3.05) and (3.06), the integrand $\Psi(t, y)$ in (3.14) involves integrals of the form $\int \sqrt{\{p(y)-t\}}\, dy$ for different t and for different limits of integration, including the complex ones. In order to facilitate

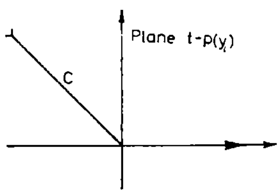

FIG. 3. The contour C in the complex $[t—p(y)]$ — plane.

the evaluation of these integrals, we have taken for the modified refractive index $M(h)$ the hyperbolic law (6.23) of Chapter 15. From this the function $p(y)$ is obtained as

$$p(y) = p(y_i)+\frac{(y-y_i)^2}{y+y_l} \tag{3.18}$$

Relation (3.11) implies that

$$p(y_i) = y_l+2y_i \tag{3.19}$$

Thus formula (3.18) contains two parameters, y_i and y_l where y_i is the non-dimensional height of the inversion point. It is also expedient to introduce the special notation

$$Y = y_l+y_i \tag{3.20}$$

then

$$p''(y_i) = \frac{2}{Y} \tag{3.21}$$

We remark that in the case of the hyperbolic law (3.18) equation (3.04) is a quadratic equation with two roots y_1 and y_2 which coincide for $t= p(y_i)$.

The integrals which we need can be expressed, in the case of the hyperbolic law, in terms of elliptic integrals of the first and second kinds. However, in the cases we considered, it appeared to be more convenient to evaluate these integrals by expansion in powers of the parameter a^2, where

$$a^2 = \frac{t-p(y_i)}{4Y} \tag{3.22}$$

These expansions contain also logarithmic terms. Since the principal part of the integration over C corresponds to very small values of the parameter a^2, it is sufficient to take several of the first terms in these expansions. For large values of the parameter Y (see the beginning of Section 5) subsequent terms of the expansion are essential only on those parts of the contour where the whole integrand is already small.

In conclusion, we will consider the analytic continuation of the functions $F(t, y', y)$ and $\Psi(t, y)$ over the whole complex t-plane. The difficulty is that, on the one hand, the quantities $S(y)$, $S(y')$, S_0' and $\chi(\nu)$, which enter into these functions, are originally defined only on the real axis for $t<p(y_i)$, $\nu>0$ (where the arithmetic values of the radicals $\sqrt[4]{\{p(y')-t\}}$ and $\sqrt[4]{\{p(y)-t\}}$ are taken) and on the other hand the values of the integrand for $t<p(y_i)$ are sufficient only for calculations using the reflection formula (see Section 4). In order to calculate the contour integrals, the integrand must be known for complex t, and this is achieved by means of analytic continuation.

Here, it must be kept in mind that at the point $t=p(y_i)$ the exact functions $F(t, y', y)$ and $\Psi(t, y)$ have no singularities. The asymptotic expression (3.15) for the function $\Psi(t, y)$ has however a branch point at $t=p(y_i)$ (for the expressions $\sqrt[4]{\{p(y)-t\}}$ and $S(y)$) and at $t=p(0)$ (for the expressions $S(y)$ and ξ_0). These singularities arise because asymptotic expressions are used; there are no actual branch points, since the exact integrand must be meromorphic. Consequently, when performing the contour integration we bypass the "apparent singular points" from below taking, for example, for $t>p(y)$ the arc $[p(y_i)-t]=\pi$ and $\sqrt{[p(y)-t]} = i\sqrt{[t-p(y)]}$ where $\sqrt{[t-p(y)]}>0$. This bypass is strictly speaking conditional since formula (3.02) is not applicable for $t>p(y)$ because of the so-called Stokes phenomenon. This phenomenon can be neglected only when the section $t>p(y)$ of the integration path gives a small contribution to the value of the contour integral, which occurs in the cases we consider. The check calculations made with parabolic cylinder functions (see Chapter 15, Section 3), which give a more exact asymptotic representation of the integrand

$\Psi(t, y)$, confirm not only the qualitative but also the quantitative validity of the results obtained by using (3.15).

The function $\Psi(t, y)$ also has poles corresponding to the roots of equation (6.01) (Section 6). When the poles approach close to the contour of integration, they must be bypassed from below.

4. *Reflection Formula*

For the evaluation of the attenuation factor in the illuminated region it is natural to use the method of stationary phase since this method gives the transition to the laws of geometrical optics which are applicable far enough from the horizon. The method of stationary phase can be applied to the integral (3.14) as follows. On the real axis the function Ψ in the integrand can be written as

$$\Psi = \frac{i}{2} \frac{e^{i\Omega(t)} - e^{i\Phi(t)}}{\sqrt[4]{[p(y)-t]}\,|\chi(\nu)|\,(1-\Lambda)} \tag{4.01}$$

where

$$\left.\begin{aligned} \Omega(t) &= \xi_0 - S(y) + 2S_0 - \operatorname{arc}\chi(\nu) \\ \Phi(t) &= \xi_0 + S(y) + 2S_0 - \operatorname{arc}\chi(\nu) = \Omega(t) + 2S(y) \end{aligned}\right\} \tag{4.02}$$

$$-\operatorname{arc}\chi(\nu) = \nu\ln\nu - \nu + \operatorname{arc}\Gamma\left(\frac{1}{2} - i\nu\right) \tag{4.03}$$

and

$$\Lambda = \frac{i}{\chi(\nu)} e^{-\pi\nu + 2iS_0} \tag{4.04}$$

For the total integrand in (3.14) the following expression can be written for real t:

$$e^{i\zeta t}\Psi = \frac{i}{2} \frac{e^{i\omega(t)} - e^{i\varphi(t)}}{\sqrt[4]{[p(y)-t]}\,|\chi(\nu)|\,(1-\Lambda)} \tag{4.05}$$

where

$$\left.\begin{aligned} \omega(t) &= \zeta t + \Omega(t) \\ \varphi(t) &= \zeta t + \Phi(t) \end{aligned}\right\} \tag{4.06}$$

Since ν is also real here, we have

$$|\chi(\nu)| = \sqrt{\{1 + e^{-2\pi\nu}\}} \tag{4.07}$$

and if $\nu > 0$, then

$$|\Lambda| = \frac{1}{\sqrt{\{1 + e^{2\pi\nu}\}}} \tag{4.08}$$

The last formula shows that for $\nu > 0$ $(t < p(y_i))$ the absolute value of $\varLambda$ is less than unity $\left(\text{in particular } |\varLambda| = \frac{1}{\sqrt{2}} \text{ for } \nu = 0\right)$, and it tends rapidly to zero as ν increases. Consequently, if we have to find the stationary phase point at $t < p(y)$, we can neglect the phase of the denominator $1 - \varLambda$. Then the stationary phase points t_1 and t_2 of the first and the second components on the right-hand side of (4.05) are obtained from the equations

$$\omega'(t_1) = 0; \qquad \varphi'(t_2) = 0 \tag{4.09}$$

or

$$\zeta = -\Omega'(t_1); \qquad \zeta = -\Phi'(t_2) \tag{4.10}$$

For given ζ and y the values of t_1 and t_2 are different.

Calculations show that the functions $-\Omega'(t)$ and $-\Phi'(t)$ have a maximum. Consequently, we find two values of t_1 and two values of t_2 (at least, if ζ is not too large). Only values of t_1 and t_2 should be taken for which $t < p(y_i)$ $(a^2 < 0)$ because the phase of the denominator $1 - \varLambda$ can be neglected when determining the stationary phase points only for these values.

Having found the points t_1 and t_2, we can evaluate (3.14) by applying the method of stationary phase to each component of (4.05) separately. In this way we come to the reflection formula for the attenuation factor V_1

$$V_1(\zeta, y) = \frac{e^{i\omega(t_1)}}{\sqrt[4]{[p(y) - t_1]}} \frac{A(t_1)}{\sqrt{[-2\omega''(t_1)]}} - \frac{e^{i\varphi(t_2)}}{\sqrt[4]{[p(y) - t_2]}} \frac{A(t_2)}{\sqrt{[-2\varphi''(t_2)]}} \tag{4.11}$$

where

$$A(t) = \frac{1}{|\chi(\nu)|(1 - \varLambda)} \tag{4.12}$$

The first term of the reflection formula (4.11) is the direct wave, the second term is the wave reflected from the earth. This formula has the same structure as the usual reflection formula of geometrical optics; it includes, however, corrections found by the exact analysis of a wave passing through the layer adjoining the inversion point.

Let us note that as ζ decreases t_1 and t_2 decrease and the corresponding values of ν increase. For positive sufficiently large values of ν we may put

$$\varLambda = 0; \qquad \chi(\nu) = 1; \qquad A = 1 \tag{4.13}$$

and, consequently, more simple expressions for the functions $\Omega(t)$ and $\Phi(t)$ can be used, namely

$$\left.\begin{aligned} \Omega(t) &= \Xi_0 - S(y) \\ \Phi(t) &= \Xi_0 + S(y) \end{aligned}\right\} \tag{4.14}$$

where

$$\Xi_0 = \xi_0 + 2S_0 = \lim_{y' \to \infty} \left[S(y') - \frac{2}{3} y'^{3/2} + t\sqrt{y'} \right] \tag{4.15}$$

With these simplifications the reflection formula (4.11) goes over into the usual reflection formula which follows from the laws of geometrical optics in an inhomogeneous atmosphere. Therefore, the latter is applicable to rays sufficiently distant from the limiting rays QO and $QO''O'$ on Fig. 1b, or more exactly, to those rays for which $\nu(t_1)$ and $\nu(t_2)$ are large positive numbers. As it is easy to see, for the limiting rays themselves we have $\nu=0$ and geometrical optics is not applicable for these rays.

Returning to the general reflection formula (4.11), let us introduce the following notation for the maximum values of $-\Omega'(t)$ and $-\Phi'(t)$

$$\zeta_1 = [-\Omega'(t)]_{\max}; \qquad \zeta_2 = [-\Phi'(t)]_{\max} \tag{4.16}$$

Because of the formulae (4.02), the following inequality is always satisfied:

$$\zeta_1 < \zeta_2 \tag{4.17}$$

Hence, we see that the points t_1 and t_2 of the stationary phase exist for both components in (4.05) only if $\zeta < \zeta_1$. For $\zeta > \zeta_1$ the equation $\omega'(t)=0$ has no real solution and the direct wave is not expressed by the first component of (4.11). Consequently, the value $\zeta = \zeta_1$ determines the horizon of the direct waves (see Section 2). Similarly, the value $\zeta = \zeta_2$ determines the horizon of the waves reflected from the earth.

The physical meaning of ζ_2 is that the electromagnetic waves penetrate in the region where $\zeta > \zeta_2$ only because of diffraction, therefore, $\zeta = \zeta_2$ is the boundary of the shadow region. The physical meaning of ζ_1 is (since for $\zeta < \zeta_1$, the reflection formula (4.11) is applicable) that $\zeta = \zeta_1$ is the boundary of the illuminated region. The region $\zeta_1 < \zeta < \zeta_2$ is the intermediate region between both horizons.

Since the maximum values of the functions $-\Omega(t)$ and $-\Phi(t)$ are attained near the point $t = p(y_i)$, the quantities

$$\tilde{\zeta}_1 = -[\Omega'(t)]_{t=p(y_i)}; \qquad \tilde{\zeta}_2 = [-\Phi'(t)]_{t=p(y_i)} \tag{4.18}$$

will be very close to the quantities determined from (4.16), as we will show by examples in Section 5. Consequently, the position of the horizons can be determined approximately by formulae of the type of (4.18) which is much more simple than to construct the graphs of the functions $-\Omega'(t)$ and $-\Phi'(t)$ required for the use of (4.16). For the hyperbolic

law (3.18) the formulae (4.18) reduce to

$$\tilde{\zeta}_1 = G_0 - G(y); \qquad \tilde{\zeta}_2 = G_0 + G(y) \tag{4.19}$$

where

$$G_0 = \sqrt{y_l} - \frac{\sqrt{Y}}{2} \cdot \ln \frac{\sqrt{Y}+\sqrt{y_l}}{\sqrt{Y}-\sqrt{y_l}} + \frac{\sqrt{Y}}{2}\left[C_1 + \frac{1}{2}\ln(Y^3)\right] \tag{4.20}$$

$$G(y) = -\sqrt{(y_l+y)} + \sqrt{y_l} + \frac{\sqrt{Y}}{2}\left[\ln \frac{\sqrt{Y}+\sqrt{(y_l+y)}}{\sqrt{Y}-\sqrt{(y_l+y)}} - \ln \frac{\sqrt{Y}+\sqrt{y_l}}{\sqrt{Y}-\sqrt{y_l}}\right] \tag{4.21}$$

and

$$C_1 = C + 7\ln 2 - 4 = 1{\cdot}429 \tag{4.22}$$

(C is Euler's constant).

When the transformation is made to the usual (dimensional) coordinates the second formula (4.19) reduces to the formula for the horizon distance of the reflected waves given in our previous paper (Equation (6.32) of Chapter 15). As we already said, the first formula (4.19) determines the horizon distance of the direct wave.

In conclusion, we note that the reflection formula (4.11) is applicable to the calculation of the attenuation factor V_1 almost up to the direct wave horizon Γ_1 itself.

5. *Numerical Results in Non-dimensional Coordinates*

When calculating the attenuation factor V_1 for a hyperbolic inversion law, we chose the following numerical values of the parameters which enter into the function $p(y)$ (formulae (3.18)–(3.20)):

TABLE 1

No.	y_i	y_l	Y	$p(y_i)$	$p(0)-p(y_i)$	y
1	10·40	197·61	208·01	218·41	0·542	2·08
2	5	95	100	105	0·260	1
3	2·40	45·67	48·07	50·48	0·125	0·48
4	1·16	21·95	23·11	24·27	0·060	0·23

The functions $p(y)$ for the given values of the parameters are shown in Fig. 4. This choice permits us to compute, for a given M-profile, the propagation of four different wavelengths, which are in the ratio 1:3:9:27,

(see Section 7). The first line of Table 1 corresponds to the shortest wave and the fourth line to the longest wave.

We took $y = y_i/5$ in all cases, i.e., we assumed the height of one of the corresponding points to be equal to one-fifth the height of the inversion

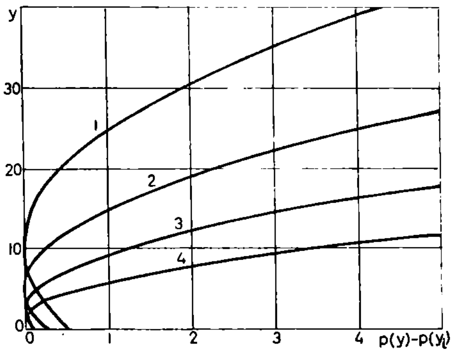

FIG. 4. Graphs of the functions $p(y)$ for values of the parameters from Table 1.

layer. The other point was taken at a great height above the inversion layer — sufficiently great to allow the use of expression (3.13) for the attenuation factor $V_1(\zeta, y)$.

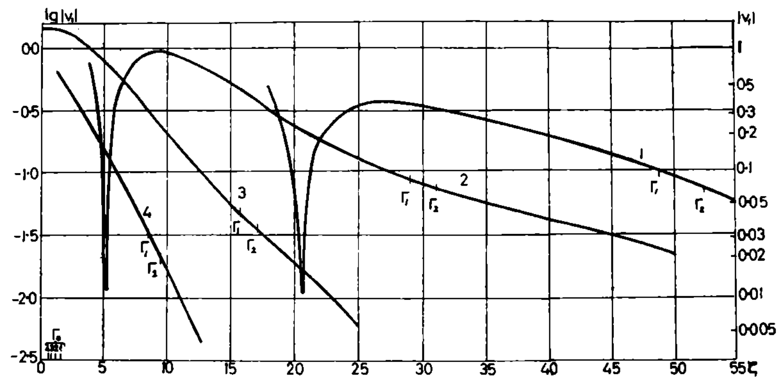

FIG. 5. Dependence of the attenuation factor V_1 on ζ. The $N^0 N^0$ 1, 2, 3, 4 of the curves correspond to the $N^0 N^0$ of the lines in Table 1 and to the $N^0 N^0$ of the curves on Fig. 4.

The four computed curves of the attenuation factor V_1, as a function of the variable ζ, are given in Fig. 5. The subscripts 1, 2, 3, and 4 on the curves show to which line of Table 1 and to which p-curve of Fig. 4 the

given curve for the attenuation factor correspond. On each curve the point Γ_1 marks the position of the direct wave horizon and the point Γ_2 marks the position of the horizon for waves reflected from the earth. The points Γ_0 near the origin, which are given the same indices 1, 2, 3, and 4, determine the horizon (the line-of-sight boundary) for a homogeneous atmosphere; the corresponding values ζ_0 are obtained from the simple formula $\zeta_0=\sqrt{y}$.

As is seen, long-range propagation takes place in all four of the cases considered, and is most markedly shown in curve 1 as would be expected. The phenomenon of long-range propagation attenuates continuously for curves 2, 3, and 4. Even for the curve 4, however, the value of the attenuation factor for $\zeta\approx 5$ is of the order $|V|\approx 0{,}1$ while in a homogeneous atmosphere V_1 assumes a value four orders lower $(|V_1|\approx 0{\cdot}000013)$ for the same ζ and y.

The values of the function $|V_1|$ at the horizons Γ_1 and Γ_2 are given in Table 2. From this table it is seen that the values of the attenuation factor at both horizons Γ_1 and Γ_2 vary within rather wide limits, about 3 or 3·5 times. The values of $|V_1|$ at the Γ_0 horizon for normal propagation and for the same values of y are also given in Table 2. A comparison of the columns shows that, owing to the dependence on y, the values of the attenuation factor at the horizon for normal propagation have approximately the same dispersion as for anomalous propagation at the Γ_1 and Γ_2 horizons.

TABLE 2

No.	Γ_1	Γ_2	Γ_0	y
1	0·096	0·070	0·24	2·08
2	0·095	0·080	0·19	1
3	0·047	0·035	0·14	0·48
4	0·031	0·023	0·083	0·23

It can be noted that at the Γ_1 and Γ_2 horizons no sudden variation in the character of the propagation occurs: the attenuation factor begins to decrease monotonically in the illuminated region, to the left of both the horizons. This leads in particular to the attenuation factor in the case of anomalous propagation being 2–4 times less at the Γ_1 and Γ_2 horizons than for normal propagation at the Γ_0 horizon (Table 2). Such a behaviour of the attenuation factor can probably be explained by the

fact that not only in the region beyond the Γ_1 and Γ_2 horizons but also to the left of these horizons, an essential part is played by diffraction (more accurately, wave) phenomena, taken into account by the reflection formula (4.11) and not included in the laws of geometrical optics. In order to elucidate whether the simple formulae (4.18)–(4.22) can be applied for the computation of the distances of the Γ_1 and Γ_2 horizons, let us compare the results which they give in the cases we considered with the results obtained from formula (4.16).

TABLE 3

No.	ζ_1	$\tilde{\zeta}_1$	ζ_2	$\tilde{\zeta}_2$
1	49·11	49·11	52·26	52·26
2	28·56	28·56	30·74	30·74
3	16·08	15·99	17·52	17·50
4	8·67	8·45	9·52	9·50

Table 3 shows that both formulae agree closely. Consequently, the simple formulae of Chapter 15 can be used in practice for the computation of the distance to the horizons.

6. *Attenuation Factor in Deep Shadow. Residue Series*

For the investigation of the attenuation factor in deep shadow it is convenient to use the residue series which is obtained from the integral (3.14) by the usual method (see Chapter 14, Section 6). In order to obtain the residue series, it is first necessary to find the exact position of the poles of the function $\Psi(t, y)$ i.e., of the roots of the equation

$$1 - \Lambda = 0 \tag{6.01}$$

These roots lie near the contour C (Fig. 3) or within it. If we denote by

$$\Delta t = t - p(y_i) \tag{6.02}$$

then the values of Δt for the roots found can be obtained from Table 4 where the first column indicates the case considered (the number of the line in Table 1) and the second column shows the number of the root.

The position of the real parts of the first three roots with respect to the $p(y)$ curve for the first case is shown in Fig. 6. We see that only the first root corresponds to the "trapped" wave in the usual sense; the other two roots give waves which would, from the geometrical optics standpoint,

easily escape beyond the limits of the inversion layer. However, these "leakage" waves have a weak attenuation and participate actively in the long-range propagation process. Let us recall that for normal propagation

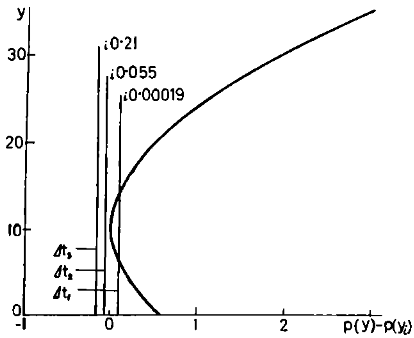

FIG. 6. Roots t_m corresponding to the trapped and to the non-trapped waves.

$t_1 = 1{\cdot}17+i2{\cdot}02$ so that even the third wave of Table 4 attenuates ten times more slowly than the least attenuated wave under normal propagation conditions. For the rest of the cases all the roots correspond to the "leakage" waves.

TABLE 4

N	m	Δt_m
1	1	$0{\cdot}10653+i0{\cdot}00019$
	2	$-0{\cdot}06364+i0{\cdot}05523$
	3	$-0{\cdot}1633\ +i0{\cdot}2107$
	4	$-0{\cdot}2495\ +i0{\cdot}3913$
2	1	$-0{\cdot}06338+i0{\cdot}06518$
	2	$-0{\cdot}1733\ +i0{\cdot}3293$
3	1	$-0{\cdot}1038\ +i0{\cdot}2238$
	2	$-0{\cdot}1883\ +i0{\cdot}6934$
4	1	$-0{\cdot}0852\ +i0{\cdot}4661$
	2	$-0{\cdot}1275\ +i1{\cdot}1318$

Let us transform equation (6.01) to a simple approximate form which will allow comparison of our theory with other theories of long-range propagation. We begin with the "trapped" waves which have almost real t between $p(y_i)$ and $p(0)$ (such as the first root in Table 4) and, consequently have negative values of ν. For $\nu < 0$, we put

$$\nu = (-\nu)\,e^{i\pi}; \qquad \ln \nu = \ln(-\nu)+i\pi \tag{6.03}$$

Then we have in addition to (3.10)

$$\chi(\nu) \to 1 \qquad \text{for } \nu \to -\infty \tag{6.04}$$

and we obtain from (3.05)

$$S_0 = S_1 - \frac{i\pi}{2}\nu \tag{6.05}$$

where

$$S_1 = \int_0^{y_1} \sqrt{\{p(y) - t\}}\, dy \tag{6.06}$$

and y_1 denotes the least positive root of equation (3.04). Taking these formulae into account equation (6.01) becomes

$$i\, e^{2iS_1} = \chi(\nu) \tag{6.07}$$

If ν is large and negative (strongly trapped waves) then because of the relation (6.04) we obtain the following more simple equation

$$S_1 = \left(m - \frac{1}{4}\right)\pi \qquad m = 1, 2, \ldots \tag{6.08}$$

which corresponds to the known characteristic equation of trapped waves.

Now, let us suppose that ν is positive or complex with a positive real part, i.e., $\mathrm{Re}\, t < p(y_i)$ or $\mathrm{Re}\, \Delta t < 0$. In this case, it is not possible to determine the quantity S_1 by using equation (6.06), as it is not even known which of the complex roots y_1 or y_2 should be taken in (6.06). However, inverting (6.05), we can always determine S_1 by using the relation

$$S_1 = S_0 + \frac{i\pi}{2}\nu \tag{6.09}$$

and then from $\Lambda = 1$ we again obtain (6.07). For an appropriate choice of arc ν in (6.03) we will always have $\chi(\nu) \to 1$ for $|\nu| \to \infty$ with the exception of arc $\nu = -\pi/2$. Since we have $\chi(0) = \sqrt{2}$ we can take as a first, quite rough approximation $\chi(\nu) = 1$ even for the "leakage" wave and then we again obtain equation (6.08).

We note that the simplified equation (6.08) can be used also for normal propagation when we must put in equation (6.06) $p(y) = y$ and $y_1 = t$. We thus obtain from (6.08)

$$t_m = \left[\frac{3}{2}\left(m - \frac{1}{4}\right)\pi\right]^{2/3} e^{i\frac{\pi}{3}} \tag{6.10}$$

which correspond approximately to the roots of the characteristic equation $w(t_m) = 0$ for the homogeneous atmosphere.

In order to check equation (6.08), we calculate by means of (6.09) the value of S_1 for the roots which we have found and obtain the following results.

TABLE 5

No.	m	$\left(m-\frac{1}{4}\right)\pi$	S_1	ν
1	1	2·356	2·326−i0·001	−0·768−i0·0014
	2	5·498	5·537+i0·047	0·459−i0·398
	3	8·639	8·646+i0·009	1·178−i1·519
	4	11·781	11·784+i0·005	1·800−i2·821
2	1	2·356	2·444+i0·062	0·317−i0·376
	2	5·498	5·501+i0·011	0·867−i1·646
3	1	2·356	2·315+i0·011	0·360−i0·776
	2	5·498	5·516+i0·014	0·656−i2·402
4	1	2·356	2·436+i0·079	0·207−i1·120
	2	5·498	5·499+i0·007	0·319−i2·718

We see that, by calculating S_1 for the roots found, we can ascribe to every root a definite number m using the approximate relation (6.09).

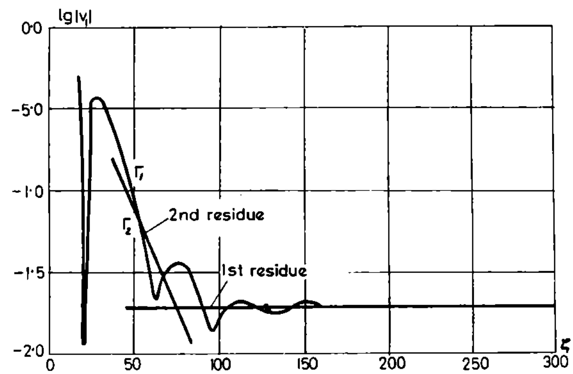

FIG. 7. Dependence of the attenuation factor V_1 on ζ in deep shadow; calculated by the residue series.

In Fig. 7 the attenuation factor in deep shadow for the first case is shown calculated by using the residue series. Figure 7 shows that the first term of the residue series, which corresponds to the pole t_1, determines

the attenuation factor only for $\zeta > 150$ i.e., if $\lambda = 1$ cm, for $s > 1000$ km. Since the first term has a negligible attenuation, the absolute value of the attenuation factor at such far distances will be almost constant; the asymptote on Fig. 7 is almost horizontal. The attenuation factor approaches the asymptote in an oscillatory manner (in the deep shadow on Fig. 7). These oscillations arise from the interference of the first and second "simple waves".

Thus the first simple wave which has the least attenuation is excited very slightly if the excitation comes from a wave incident from above on the tropospheric waveguide; consequently this simple wave can have importance only at very far distances. Near the Γ_1 and Γ_2 horizons the second and, in part, the third term of the residue series are most important. This phenomenon must have a general character since if the simple wave is "trapped" (see above) and does not or almost not leak out of the inversion layer (which is the cause of its negligible attenuation) then from reciprocity considerations it follows that this wave is but feebly excited by radiators above the inversion layer. Waves with stronger attenuation penetrate the space above the inversion layer to a larger degree; consequently, they are excited more strongly and play a fundamental part near the horizons.

Because of this circumstance, the Γ_1 and Γ_2 horizons actually determine in a certain sense the range of radiowave propagation even for strongly pronounced superrefraction, as is seen from Fig. 7.

Long-range propagation is usually analysed on the basis of the residue series and it is assumed that only trapped waves can have low attenuation ($\mathrm{Re}\,\Delta t_m > 0$). Actually, however, leaking waves (with $\mathrm{Re}\,\Delta t_m < 0$) also attenuate feebly in some cases. Consequently, waves which are several times longer than the "critical" wavelength λ_0 defined according to Bremmer [25] can still contribute to long-range propagation in a tropospheric waveguide.

Our computations showed that the attenuation factor can be calculated with a small number of the first terms of the residue series, so that it almost joins the reflection formula. Calculations by means of numerical integration thus become unnecessary (see Section 3).

7. *Numerical Example*

In order to facilitate the physical analysis of the numerical results which we obtained in Section 5, let us consider a concrete case.

We take as an example the M-profile shown on Fig. 8 and compute the attenuation factor V_1 for the following wavelengths: (1) 3·33 cm;

(2) 10 cm; (3) 30 cm; (4) 90 cm. The graph of V_1 is given in Fig. 9. The numbers on the curves of Fig. 9 indicate the wavelengths enumerated above. The horizontal scale (abscissae) is the range ζ in dimensionless units. The left vertical scale (ordinates) is log V_1 (to the base 10) and the right-hand scale gives values of $|V_1|$.

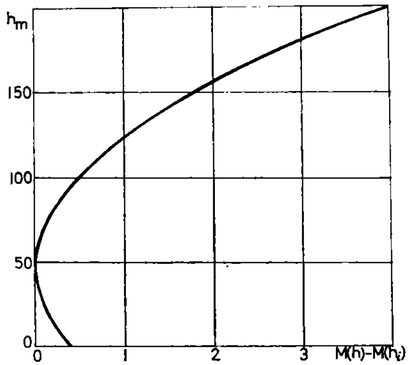

FIG. 8. Dependence of h on M (M-profile) $h_i = 46{\cdot}5$ m. $l = 884{\cdot}0$ m. $H = 930{\cdot}5$; $M(h_i) = 153{\cdot}5$; $M(O) - M(h_i) = 0{\cdot}381$.

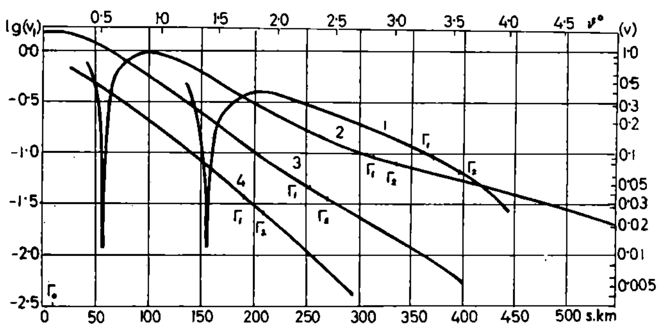

FIG. 9. Dependence of the attenuation factor V_1 on the distance for wavelengths: (1). 3·33 cm (2). 10 cm (3). 30 cm (4). 90 cm.

The dispersion was not taken into account in our computations. It was assumed that the M-curve has the same shape for all four wavelengths for which the attenuation factor V_1 is given in Fig. 9.

The M-curve in Fig. 8 is constructed according to the hyperbolic law

$$M(h) = M(h_i) + \frac{1}{a}\,\frac{(h-h_i)^2}{h+l} \tag{7.01}$$

in which

$$M(h_i) = \frac{l+2h_i}{a} \tag{7.02}$$

The hyperbolic law involves two parameters, namely h_i and l. They have the dimensions of height and are connected with the dimensionless constants y_i and y_l in equation (3.18) by the relations

$$y_i = \frac{kh_i}{m}; \qquad y_l = \frac{kl}{m}; \qquad m = \left(\frac{ka}{2}\right)^{1/3} \tag{7.03}$$

in which h_i is the height of the inversion point or, what is the same, the height of the atmospheric waveguide. As is easily shown, the height

$$H = h_i + l \qquad\qquad \left(Y = \frac{kH}{m}\right) \tag{7.04}$$

determines the radius of curvature of the M-curve at the inversion point.

The horizon Γ_0 for propagation in a homogeneous atmosphere is also marked off along the horizontal axis of Fig. 9. This horizon is determined by the height of the observation point h and is independent of the wavelength $\left(\text{we have taken } h = \frac{1}{5} h_i \text{ everywhere}\right)$. The point Γ_1 on each curve determines the position of the direct wave horizon and the point Γ_2 determines the position of the horizon for waves reflected from the earth (Sections 2 and 4). The Γ_1 and Γ_2 horizons vary as the wavelengths vary and, consequently, for each curve they are different.

In all cases one can see the phenomenon of long-range radiowave propagation, attenuating as the wavelength increases. Taking into account the variation of the wavelength from one curve to another (the wavelengths are in the ratio, 1:3:9:27), it should be noticed that near the horizon the dependence of the attenuation factor on the wavelength is comparatively slight.

The wavelength enters into the formula for the horizon distance (see Chapter 15, Section 6) only through the logarithm. Consequently, the distances of the horizons are in an arithmetic progression if the wavelengths, as in Fig. 9, form a geometric progression. The values of the attenuation factor at both the Γ_1 and the Γ_2 horizons depend on the wavelength to the same degree as for normal radiowave propagation (see Table 2).

Because of these circumstances, the identification of the range of radiowave propagation as the distance of the horizon for the direct and the reflected wave must be made with some reservation. The range of propa-

gation can be defined, for example, as that distance at which the attenuation factor has the absolute value 0·1, the values of the attenuation factor at longer distances being still less. With this latter definition, the "range of propagation" lies between the distances of the Γ_1 and Γ_2 horizons for the Curve 1 on Fig. 9 while for the other curves this distance is less than the distance Γ_1. As seen from the figures these four distances also form an arithmetic progression (in a rough approximation). We note that in order to estimate the distance of propagation according to the 0·1 criterion it is usually sufficient to apply the reflection formula of Section 4 and in some cases to extrapolate the curves thus obtained.

The direct purpose of this paper (see Section 1) was to check the formulae for the distance of radiowave propagation. We have shown above that a simple and intuitive picture of long-range radiowave propagation in the presence of an inversion layer can be obtained by introducing the horizons of the direct and the reflected wave. However, the distance of propagation can be identified with the distance of one of the horizons in a crude approximation only. The decrease in the attenuation factor (after the oscillations peculiar to the illuminated region terminate) begins at distances smaller than the first horizon. Consequently, as shown in Section 5, the values of the attenuation factor V_1 on the Γ_1 and Γ_2 horizons are 2–4 times less than the values on the ordinary Γ_0 horizon in case of propagation in a homogeneous atmosphere. Moreover, the attenuation factor obviously decreases near the Γ_1 and Γ_2 horizons much more slowly than for normal propagation.

All these considerations show that the Γ_1 and Γ_2 horizons introduced in the case of anomalous propagation characterize the range of radiowave propagation more roughly than does the Γ_0 horizon under normal propagation. Nevertheless, there is no doubt that the Γ_1 and Γ_2 horizons can be effectively used for an approximate estimate of the distance of propagation; this is seen for instance from a comparison of the attenuation factors near these horizons and in deep shadow in Fig. 7.

It should be stressed that the M-profile we chose has a very feeble inversion: the difference $M(0)-M(h_i)$ does not exceed several tenths. Such an inversion can in practice remain undetected. However, our calculations show that even such a weak M-profile radically alters the character of radiowave propagation and leads to long-range propagation.

CHAPTER 17

RADIOWAVE PROPAGATION ALONG A TROPOSPHERIC WAVEGUIDE (DUCT) NEAR THE EARTH†

Abstract — This paper deals with the theory of radiowave propagation between corresponding points in an inversion layer adjacent to the earth ("tropospheric waveguide" or "duct"). Methods are developed to evaluate the attenuation factor in the presence of superrefraction. Several specific propagation cases are treated numerically. The criterion determining the range of propagation and its dependence on the properties of the inversion layer and wavelength is stated more precisely.

Introduction

In the present paper the theory of radiowave propagation in a tropospheric waveguide (inversion layer) adjacent to the earth is developed assuming that the two corresponding points are both within the waveguide. Such radiowave propagation can be called intra-layer or intra-waveguide in contrast to the case considered in Chapter 16, when one of the corresponding points was supposed to be high above the inversion layer.

Radiowave propagation in a tropospheric waveguide is considered in a number of theoretical papers (for example, see Refs. 23, 24, 25), there are, however, still many obscure points in this problem. In the present work, we investigate intra-layer propagation by developing the methods described in Chapters 15 and 16.

1. *Basic Formulae*

In this Section, we will write down the fundamental formulae obtained in Chapters 14, 15 and 16.

The attenuation factor V for an arbitrary dependence of the refractive index of the atmosphere on the height is determined by the contour integral

$$V(x, y, y', q) = \sqrt{\left(\frac{x}{\pi}\right)}\, e^{-i\frac{\pi}{4}} \int_C e^{ixt} F(t, y, y', q)\, dt \qquad (1.01)$$

† Fock, Wainstein and Belkina, 1958.

where the contour C described in a positive direction encloses all the poles of the integrand (Chapter 14).

We will confine ourselves to the value $q=\infty$, which corresponds to an arbitrary polarization at decimeter and shorter wavelengths and to a horizontal polarization at longer wavelengths. In this case, the function $F(t, y, y', \infty)=F(t, y, y')$ has the form (for $y'>y$)

$$F(t, y, y') = -\frac{1}{2i} f_1(y', t)\left[f_2(y, t)-\frac{f_2(0, t)}{f_1(0, t)} f_1(y, t)\right] \tag{1.02}$$

y and y' are here the non-dimensional heights of the corresponding points

$$y = \frac{kh}{m}; \qquad y' = \frac{kh'}{m}; \tag{1.03}$$

x is the non-dimensional distance between them along the surface of the earth:

$$x = \frac{ks}{2m^2} \tag{1.04}$$

and the parameter m has the value

$$m = \left(\frac{ka}{2}\right)^{1/3} \tag{1.05}$$

where a is the radius of the earth. The functions f_1 and f_2 are independent solutions of the differential equation

$$\frac{d^2f}{dy^2}+[p(y)-t]f = 0 \tag{1.06}$$

whose Wronskian equals $-2i$.
The function $p(y)$ is connected with the modified refractive index $M(h)$ by the relation

$$p(y) = \frac{2m^2}{10^6} M(h) = 2m^2\left(n-1+\frac{h}{a}\right) \tag{1.07}$$

(n is the refractive index of air) and is the non-dimensional analogue of $M(h)$.

In Chapter 15 equation (1.06) has been integrated asymptotically assuming that the function $p(y)$ has one minimum, i.e., the modified refractive index of the atmosphere has one inversion point. For example the refractive index which depends on height according to the hyperbolic law

$$M(h) = M(h_i)+\frac{1}{a}\frac{(h-h_i)^2}{h+l} 10^6 \tag{1.08}$$

(h_i is the inversion height, l is a parameter) possesses this property. To this $M(h)$ corresponds the function

$$p(y) = p(y_i) + \frac{(y-y_i)^2}{y+y_l} \tag{1.09}$$

Expression (1.09) is used in all the calculations in this Chapter and in Chapter 16.

Besides the non-dimensional coordinates x, y, y' the attenuation factor V depends on the parameters y_i and y_l or on $Y=y_i+y_l$ and y_i.

As has been shown in Chapter 15, the height factors f_1 and f_2 can be expressed by means of the parabolic cylinder functions D_n:

$$\left.\begin{aligned} f_1(y,\, t) &= c_1(\nu)\sqrt{\left(2\frac{\mathrm{d}y}{\mathrm{d}\zeta}\right)}\, g_1(\zeta) \\ f_2(y,\, t) &= c_2(\nu)\sqrt{\left(2\frac{\mathrm{d}y}{\mathrm{d}\zeta}\right)}\, g_2(\zeta) \end{aligned}\right\} \tag{1.10}$$

where

$$\left.\begin{aligned} g_1(\zeta) &= D_{i\nu-\frac{1}{2}} \cdot \left(e^{-i\frac{\pi}{4}}\zeta\right) \\ g_2(\zeta) &= D_{-i\nu-\frac{1}{2}} \cdot \left(e^{i\frac{\pi}{4}}\zeta\right) \end{aligned}\right\} \tag{1.11}$$

The expression for $D_n(z)$ as a series is given below (formula (2.05)).

The quantity ν is defined by the relation

$$\nu = -\frac{1}{\pi}\int_{y_1}^{y_2} \sqrt{(t-p(y))}\,\mathrm{d}y \tag{1.12}$$

(y_1 and y_2 are the roots of the equation $p(y)-t=0$). Since $p''(y_i)>0$, it can be assumed that $\nu>0$ for $t<p(y_i)$; for $t>p(y_i)$ we have obviously $\nu<0$.

The variables ζ and y are connected by the relation

$$S(y)-S_0 = \int_0^{\zeta}\sqrt{\left(\frac{1}{4}\zeta^2+\nu\right)}\,\mathrm{d}\zeta \tag{1.13}$$

where

$$S(y) = \int_0^y \sqrt{[p(y)-t]}\,\mathrm{d}y \tag{1.14}$$

$$S_0 = \frac{1}{2}\int_0^{y_1}\sqrt{[p(y)-t]}\,\mathrm{d}y + \frac{1}{2}\int_0^{y_2}\sqrt{[p(y)-t]}\,\mathrm{d}y \tag{1.15}$$

and, finally

$$\left.\begin{aligned} c_1(\nu) &= e^{-\frac{\pi\nu}{4}+i\frac{\pi}{8}}\, e^{i\left(\frac{1}{2}\nu-\frac{1}{2}\nu\ln\nu-S_0\right)} \\ c_2(\nu) &= e^{-\frac{\pi\nu}{4}-i\frac{\pi}{8}}\, e^{-i\left(\frac{1}{2}\nu-\frac{1}{2}\nu\ln\nu-S_0\right)} \end{aligned}\right\} \tag{1.16}$$

For large, positive ζ, that correspond to heights y lying above the inversion point y_i and far enough away from it we have

$$\left.\begin{aligned} f_1(y,\ t) &= \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{i(S-2S_0)} \\ f_2(y,\ t) &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{-i(S-2S_0)} \end{aligned}\right\} \tag{1.17}$$

Below the inversion point and far from it (ζ large and negative) the asymptotic representation of $f_1(y,\ t)$ and $f_2(y,\ t)$ has the form

$$\left.\begin{aligned} f_1(y,\ t) &= \chi_1(\nu)\, \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{i(S-2S_0)} + e^{-\pi\nu}\, \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{-iS} \\ f_2(y,\ t) &= \chi_2(\nu)\, \frac{e^{-i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{-i(S-2S_0)} + e^{-\pi\nu}\, \frac{e^{i\frac{\pi}{4}}}{\sqrt[4]{[p(y)-t]}}\, e^{iS} \end{aligned}\right\} \tag{1.18}$$

where

$$\left.\begin{aligned} \chi_1(\nu) &= \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2}-i\nu\right)}\, e^{-\frac{\pi\nu}{2}+i(\nu-\nu\ln\nu)} \\ \chi_2(\nu) &= \frac{\sqrt{(2\pi)}}{\Gamma\left(\frac{1}{2}+i\nu\right)}\, e^{-\frac{\pi\nu}{2}-i(\nu-\nu\ln\nu)} \end{aligned}\right\} \tag{1.19}$$

Inserting the expressions (1.18) into (1.02), we obtain for the inter-layer propagation case considered ($y \leqslant y' < y_i$) the function $F(t,\ y,\ y')$ in the form

$$F(t, y, y') = \frac{i}{2}\, \frac{1}{\sqrt[4]{[p(y)-t]}\,\sqrt[4]{[p(y')-t]}}\, \frac{\left[e^{iS(y')}-\Lambda\, e^{-iS(y')}\right]\left[e^{-iS(y)}-e^{iS(y)}\right]}{1-\Lambda} \tag{1.20}$$

where

$$\Lambda = \frac{i}{\chi_1(\nu)} e^{-\pi\nu+2iS_0} \tag{1.21}$$

The attenuation factor V was calculated from the residue series

$$V(x, y, y') = 2\sqrt{(\pi x)}\, e^{i\frac{\pi}{4}} \sum_{m=1}^{\infty} R_m\, e^{ixt_m} \tag{1.22}$$

where R_m is the residue of the function $F(t, y, y')$ at the mth pole t_m. If the approximate formula (1.20) is used, then t_m is the mth root of the equation

$$1-\Lambda = 0 \tag{1.23}$$

and the residue R_m has the form

$$R_m = 2i \frac{1}{\sqrt[4]{[p(y)-t_m]}\sqrt[4]{[p(y')-t_m]}} \frac{\sin S(y) \sin S(y')}{\frac{d}{dt}[1-\Lambda]_{t=t_m}} \tag{1.24}$$

The majority of the numerical results have been obtained by using formulae (1.22)–(1.24), in which three or four terms of the residue series have been taken into account.

2. *Calculation of the Attenuation Factor with Help of Parabolic Cylinder Functions*

The asymptotic representations (1.18) of the height factors f_1 and f_2, used for the derivation of formulae (1.22)—(1.24) are valid if ζ is large and negative or lies in a certain sector adjoining the negative real half-axis. In the latter case, the absolute value of ζ must be large. However, as is seen from Table 1, in a number of cases of interest to us, serious doubts arise as to the applicability of the asymptotic formulae since the quantity ζ happens to be of the order of unity or even less, in absolute value.

TABLE 1

$Y = y_t + y_r$	No of the pole	ν	y	ζ	Δt_m acc. to (1·23)	Δt_m acc. to (2·01)
48·07	1	0·360−i0·776	0	−1·282−i0·041	−0·104+i0·224	−0·113+i0·227
23·11	1	0·207−i1·120	0	−0·729−i0·095	−0·085+i0·466	−0·107+i0·443
23·11	2	0·319−i2·718	0	−0·725−i0·246	−0·128+i1·132	−0·125+i1·145
25·24	1	0·373−i0·658	0	−1·508−i0·057	−0·148+i0·262	−0·158+i0·269

In order to elucidate this question, we calculated the function V also by using the exact expressions for the height factors in terms of the parabolic cylinder functions (formulae (1.10), (1.11)) and not their approximate asymptotic expressions. The poles t_m of the integrand $F(t, y, y')$ are roots of the equation (see (1.02) and (1.10))

$$g_1(\zeta_0) = 0 \tag{2.01}$$

where, for brevity, the value of ζ, corresponding to $y=0$, has been denoted by ζ_0. The remaining factors in $f_1(0, t)$ do not vanish. The residue R_m is calculated from the formula

$$R_m = \frac{1}{2i} \frac{f_1(y, t_m) f_1(y', t_m) f_2(0, t_m)}{\left[\frac{\mathrm{d}}{\mathrm{d}t} f_1(0, t)\right]_{t=t_m}} \tag{2.02}$$

(see also (1.10)). The quantity ζ is found from the transcendental equation

$$\sinh 2u + 2u = \frac{2}{\nu}[S(y) - S_0] \tag{2.03}$$

which is obtained if the integral (1.13) is calculated by means of the substitution

$$\zeta = 2\sqrt{\nu} \cdot \sinh u \tag{2.04}$$

The required root of the equation is determined by the condition that for a real value of $t < p(y_i)$ the quantity ζ must be a negative real number and from continuity considerations. We used the following series for the calculation of the functions (1.11)

$$D_n(z) = -\frac{2^{-\frac{n}{2}-1}}{\Gamma(-n)} \mathrm{e}^{-\frac{1}{4}z^2} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} 2^{\frac{m}{2}} \Gamma\left(\frac{m-n}{2}\right)}{m!} z^m \tag{2.05}$$

In the first instance it is interesting to compare the roots of the equation (1.23) and of the more exact equation (2.01). In Table 1 the values are given of

$$\Delta t_m = t_m - p(y_i) \tag{2.06}$$

(see (4.02)) obtained by means of formulae (1.23) and (2.01). We observe that the quantity which is used in the actual calculations by residue series is precisely Δt_m; thus the accuracy of the results is determined by this quantity. Therefore instead of the values of the poles t_m we give in all the tables the values of Δt_m. (For the choice of the values of Y see Sec-

tion 4.) The agreement in the values of Δt_m, calculated by means of the formulae (1.23) and (2.01), especially in their imaginary parts, can be considered as quite satisfactory.

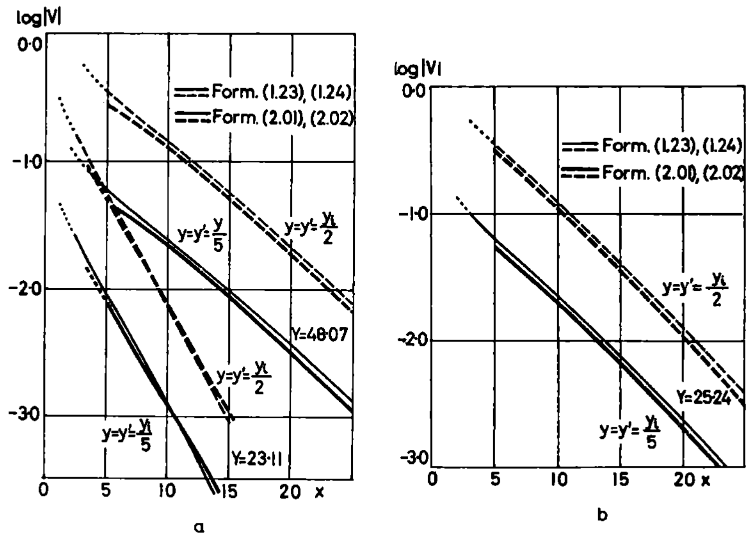

FIG. 1. The attenuation factor V calculated from formulae with parabolic cylinder functions and from asymptotic formulae. *a* — Series I, Table 3a; *b* — Series II, Table 3b.

In Figs. 1a and 1b curves are given for the attenuation factor, calculated with the asymptotic formulae (thin lines) and with the parabolic cylinder functions (thick lines). It is seen from a comparison of these curves that the results given by both methods are even closer than could be expected on the basis of Table 1, especially for $Y=23{\cdot}11$. Consequently, further discussion is based on calculations with the asymptotic formulae, as the more simple, except the case considered in Section 3.

Similar calculations have been performed for the case investigated in Chapter 16, and asymptotic formulae were found to be applicable in that case with about the same accuracy as in the case of intra-layer propagation.

3. *Calculation of Height Factors with the Help of Airy Functions*

Approximate formulae for the height factors of feebly damped "trapped" waves, for which the appropriate pole is situated between the values $p(y_l)$ and $p(0)$, have been derived in Section 8 of Chapter 14 by using the Airy functions $u(x)$ and $v(x)$. In our case, the trapped waves correspond

to the first poles for the curves with $Y = 208{\cdot}01$ and $Y = 109{\cdot}20$ (Fig. 5 and 6) and the residues at these poles can be calculated by the formulae indicated above.

It is desirable to use these formulae because of the fact that, for certain values of y, say $y = y_i/2$, the quantity $p(y) - t_1$ happens to be small for the values of Y considered (see Fig. 5 and 6), which makes the use of the asymptotic formulae of Section 1 impossible. Use of the series (2.05) here is also inadequate since in the cases considered the values of ζ are large and the series converges very slowly.

Approximate formulae for the height factors of feebly damped waves for the case $y < y_i$ have the form

$$\left.\begin{aligned} f_1(y, t) &= 2c_1(\nu)\,\mathrm{e}^{-\pi\nu}\,\psi_1(y, t) \\ f_2(y, t) &= 2c_2(\nu)\,\mathrm{e}^{-\pi\nu}\,\psi_2(y, t) \end{aligned}\right\} \tag{3.01}$$

where

$$\left.\begin{aligned} \psi_1(y, t) &= \sqrt[4]{\left(\frac{\xi_1}{t-p(y)}\right)}\left[v(\xi_1) + \frac{i}{4}\,\mathrm{e}^{2\pi\nu}\,u(\xi_1)\right] \\ \psi_2(y, t) &= \sqrt[4]{\left(\frac{\xi_1}{t-p(y)}\right)}\left[v(\xi_1) - \frac{i}{4}\,\mathrm{e}^{2\pi\nu}\,u(\xi_1)\right] \end{aligned}\right\} \tag{3.02}$$

and ξ_1 is determined from the relations

$$\left.\begin{aligned} \int_{y_1}^{y} \sqrt{[t-p(y)]}\,\mathrm{d}y &= \frac{2}{3}\,\xi_1^{3/2} \qquad (y > y_1) \\ \int_{y}^{y_1} \sqrt{[p(y)-t]}\,\mathrm{d}y &= \frac{2}{3}\,(-\xi_1)^{3/2} \qquad (y < y_1) \end{aligned}\right\} \tag{3.03}$$

in which y_1 denotes the smaller root of the equation

$$p(y) - t = 0 \tag{3.04}$$

In this case the poles of the integrand are roots of the equation

$$v(\xi_0) + \frac{i}{4}\,\mathrm{e}^{2\pi\nu}\,u(\xi_0) = 0 \tag{3.05}$$

where ξ_0 denotes the value of ξ corresponding to $y = 0$. Finally, the residue R_1 equals

$$R_1 = -\frac{2i\mathrm{e}^{-2\pi\nu}\,\psi_1(y, t_1)\psi_1(y', t_1)\psi_2(0, t_1)}{\dfrac{\mathrm{d}}{\mathrm{d}t}\,\psi_1(0, t_1)} \tag{3.06}$$

The height factors can be represented in terms of Airy functions under the condition that the quantity $\mathrm{e}^{2\pi\nu}$ is small. This condition is satisfied

TABLE 2

Y	ν	Δt_1 acc. to (1·23)	Δt_1 acc. to (3·05	y	ξ_1	R_1 acc. to (1·24)	R_1 acc. to (3·06)
208·01	$-0{\cdot}768-i0{\cdot}0^{2}2$	$0{\cdot}106+i0{\cdot}0^{3}2$	$0{\cdot}101+i0{\cdot}0^{3}2$	$\frac{1}{5}y_t$	$-1{\cdot}456+i0{\cdot}002$	$0{\cdot}169-i0{\cdot}001$	$0{\cdot}166-i0{\cdot}001$
				$\frac{1}{2}y_t$	$-0{\cdot}241+i0{\cdot}002$	$0{\cdot}342-i0{\cdot}001$	$0{\cdot}142-i0{\cdot}0^{4}4$
109·20	$-1{\cdot}851-i0{\cdot}000$	$0{\cdot}354+i0{\cdot}0^{6}4$	$0{\cdot}347+i0{\cdot}0^{6}5$	$\frac{1}{5}y_t$	$-1{\cdot}196+i0{\cdot}0^{5}2$	$0{\cdot}303+i0{\cdot}0^{5}6$	$0{\cdot}283-i0.0^{5}2.$
				$\frac{1}{2}y_t$	$0{\cdot}382+i0{\cdot}0^{5}2$	$-0{\cdot}102+i0{\cdot}293$	$0{\cdot}0762+i0{\cdot}0^{6}4$

if Reν is negative and not small ($-\text{Re}\nu \gg 1$). This latter condition is fulfilled for the poles t_1 in the cases $Y = 208{\cdot}01$ and $Y = 109{\cdot}20$ (Table 2), consequently, the given method is applicable to the calculation of the residues at these poles.

It can be shown that if ξ_1 is large and negative then the residue formula (3.06) goes over into (1.24). Thus formula (1.24) is certainly applicable if ξ_1 is a large negative number (or lies in a certain sector adjoining the negative real semi-axis and is large in absolute value). If, however, ξ_1 is small or positive, then the asymptotic formula (1.24) becomes inapplicable.

The above is confirmed by Table 2. In both cases $Y = 208{\cdot}01$ and $Y = 109{\cdot}20$ there correspond to the values $y = y_i/5$ values of ν and ξ_1 with rather large negative real parts and the residues obtained by using (1.24) and (3.06) are very close to each other. To values of $y = y_i/2$ for the same ν correspond small ξ_1 (and for $Y = 109{\cdot}20$ the real part of ξ_1 is already positive), the residues, calculated by using the formulae (1.24) and (3.06) differ considerably. In the case, the residues are to be calculated from equation (3.06) while equation (1.24) is inapplicable.

Thus, if the quantity $p(y) - t$ is small (ξ_1 small), equation (1.18) is not suitable for the calculation of the height factors and equations (3.01) to (3.03) are to be used instead. We observe that the roots of equations (3.05) and (1.23) are practically the same.

The Airy functions can also be applied to the investigation of the height of the layer in which the trapped wave is propagated. As is seen from the formulae (3.03) the real part of ξ_1, is negative only for $y < y_1$, where y_1 is the lesser root of equation (3.04). For $y > y_1$ the quantity Reξ_1, becomes positive and increases with increasing y (the quantity Imξ always remains very small since for the trapped waves the imaginary part of t_1 is extremely small). Now, the function $v(\xi_1)$ decays exponentially for large positive arguments and although the function $u(\xi_1)$ increases, it cannot become larger than $e^{-\pi\nu}$ ($\nu < 0$) (see equations (3.03) (1.12) and the asymptotic representation of the Airy function in the Appendix). Since $u(\xi_1)$ is multiplied by the factor $e^{2\pi\nu}$, the product remains small. Consequently the function, $\psi_1(y, t)$ and therefore the residue R_1 rapidly decays after the height y has passed through $y = y_1$.

By virtue of the above the non-dimensional height y_1 can be designated as the effective height of the given trapped wave corresponding to the pole t_1. In a first approximation, one can say that the field of this wave occupies the layer

$$0 < y < y_1 \tag{3.07}$$

and is negligibly small beyond this layer. Actually, the value $y = y_1$ is not

a completely definite boundary of the layer but gives the height near which a smooth, though rapid, attenuation of the height factors (which also means of the wave field) takes place.

Untrapped waves (y_1 complex) do not have the property mentioned above and their field is distributed within the whole inversion layer and even above it.

4. *Intralayer Wave Propagation (Numerical Results in Non-dimensional Coordinates)*

The attenuation factor V depends on the non-dimensional coordinates x, y, y' and, besides, on the function $p(y)$, which, in turn, is defined by the parameters y_i and y_l. It is convenient to introduce dependent parameters, characterizing the function $p(y)$

$$Y = y_i + y_l \tag{4.01}$$

$$p(y_i) = 2y_i + y_l \tag{4.02}$$

$$p(0) - p(y_i) = \frac{y_i^2}{y_l} \tag{4.03}$$

The calculations were carried out for two series of values of the parameters of the function $p(y)$, given in Tables 3a and 3b. The curves of $p(y) - p(y_i)$ are plotted on Fig. 2a (series I) and 2b (series II), where the

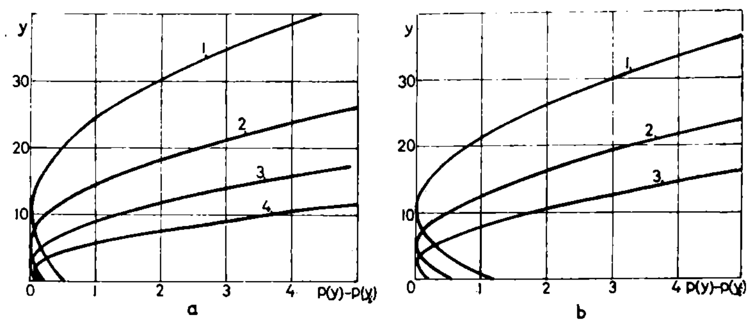

FIG. 2. Graph of the function $p(y)$: *a* — Series I, Table 3a; *b* — Series II, Table 3b.

curves are marked by numbers corresponding to those under which the appropriate parameters are given in the tables.

We have chosen parameters of the function $p(y)$ so that for a given M-profile the attenuation factor can be calculated, for four wavelengths in the ratio 1:3:9:27 (series I) or three wavelengths in the ratio 1:3:9 (series II).

TABLE 3a

Series I

No. of the curve	Y	y_t	y_l	$p(y_i)$	$p(0)-p(y_i)$
1	208·01	10·40	197·61	218·41	0·542
2	100	5	95	105	0·260
3	48·07	2·40	45·67	50·48	0·125
4	23·11	1·16	21·95	24·27	0·060

TABLE 3b

Series II

No. of the curve	Y	y_t	y_l	$p(y_i)$	$p(0)-p(y_i)$
1	109·20	10·40	98·80	119·60	1·095
2	52·5	5	47·5	57·5	0·526
3	25·24	2·40	22·84	27·6	0·252

We observe that the first series of parameters was adopted also in Chapter 16, in which the case was considered when one of the corresponding points is high above the inversion layer and the other lies within the wave-guide. We took for the latter $y=y_i/5$ where y_i is the height of the inversion point.

In the present Chapter the attenuation factor V is calculated for propagation within the wave-guide, and the transmitter and receiver heights were supposed to be equal ($y=y'$). The calculations were carried out for two cases

$$y = y' = \frac{y_i}{5} \tag{4.04}$$

$$y = y' = \frac{y_i}{2} \tag{4.05}$$

The attenuation factor $|V|$ is represented in Figs. 3 and 4 as a function of the non-dimensional distance x, the values of $\log|V|$ being plotted along the vertical axis. Figure 3 corresponds to the first and Fig. 4 to the second series. The solid curves refer to the case (4.04) and the dashed, to (4.05). The curves of the first series have been calculated by using four,

and the curves of the second series by using three terms of the residue series.

Table 4 gives the values of Δt_m (formula (2.06)), which are involved in all the calculations and which are, moreover, of interest because they characterize the position of the poles t_m relative to the curve $p(y)$. If $\mathrm{Re}\,\Delta t_m > 0$ we have a trapped wave with a very feeble attenuation.

TABLE 4

Series I

No. of the curve	Y	No. of the pole m	Δt_m
1	208·01	1	$0{\cdot}1065+i0{\cdot}0002$
		2	$-0{\cdot}0636+i0{\cdot}0552$
		3	$-0{\cdot}1633+i0{\cdot}2107$
		4	$-0{\cdot}2495+i0{\cdot}3913$
2	100	1	$-0{\cdot}0634+i0{\cdot}0652$
		2	$-0{\cdot}1733+i0{\cdot}3293$
		3	$-0{\cdot}2544+i0{\cdot}6323$
		4	$-0{\cdot}3164+i0{\cdot}9489$
3	48·07	1	$-0{\cdot}1038+i0{\cdot}2238$
		2	$-0{\cdot}1883+i0{\cdot}6934$
		3	$-0{\cdot}2289+i1{\cdot}1712$
		4	$-0{\cdot}2509+i1{\cdot}6558$
4	23·11	1	$-0{\cdot}0852+i0{\cdot}4661$
		2	$-0{\cdot}1275+i1{\cdot}1318$
		3	$-0{\cdot}1173+i1{\cdot}8499$
		4	$-0{\cdot}0549+i2{\cdot}5290$
Series II			
1	109·20	1	$0{\cdot}3541+i0{\cdot}0^{6}4$
		2	$-0{\cdot}0062+i0{\cdot}0150$
		3	$-0{\cdot}1618+i0{\cdot}1669$
2	52·5	1	$-0{\cdot}0497+i0{\cdot}0492$
		2	$-0{\cdot}2272+i0{\cdot}3689$
		3	$-0{\cdot}3529+i0{\cdot}7653$
3	25·24	1	$-0{\cdot}1484+i0{\cdot}2621$
		2	$-0{\cdot}2635+i0{\cdot}8820$
		3	$-0{\cdot}3146+i1{\cdot}5143$

There is such a pole in the first curve of each series ($Y=208{\cdot}01$ and $Y=109{\cdot}20$, see Table 4 as well as Figs. 5 and 6) and this explains the conspicuous peculiarity of the upper curves on Figs. 3 and 4. Instead of decreasing, as would seem to be natural for the attenuation factor, these curves show an increase. Indeed, as is seen from Table 4, the imaginary parts of the first poles are very small in this case, so that the exponential factor yields practically no damping (in the region of applicability of the one-term expression for the residue series, i.e. in the region where only one trapped wave remains). The growth of the function V is due to the factor $\sqrt{x}$ (see (1.22)). Its field corresponds to a cylindrical wave since it is proportional to $1/\sqrt{x}$ and not to $1/x$. A more immediate physical meaning is obtained by using the function Ψ rather than V, defined by the relation

$$V = 2\sqrt{(\pi x)}\, e^{i\frac{\pi}{4}}\, \Psi \tag{4.06}$$

from which

$$\Psi = \sum_{m=1}^{\infty} R_m\, e^{ixt_m} \tag{4.07}$$

(see (1.22)). This function is the attenuation factor of a cylindrical wave that is propagated in a tropospheric waveguide adjacent to the earth.

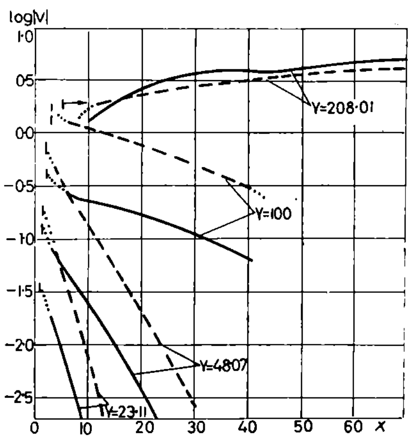

FIG. 3. Dependence of the attenuation factor V on the dimensionless distance x (Series I).

The propagation is as if there is a reflecting layer (its height being approximately equal to y_1) beyond whose limits the trapped wave does not leak (see the end of Section 2). Therefore, the amplitude R_m of this

wave as a function of y can decrease for growing y if $y > y_1$. This takes place in Fig. 3 for $Y = 208{\cdot}01$ and on Fig. 4 for $Y = 109{\cdot}20$; in all the other cases the attenuation factor increases monotonously as y increases.

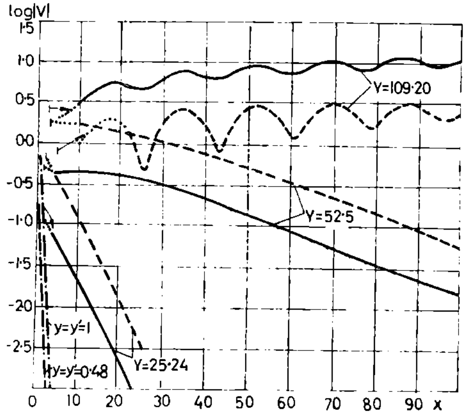

FIG. 4. Dependence of the attenuation factor V on the dimensionless distance x (Series II).

As is seen from Table 4 (see also Figs. 5 and 6), the second poles of the curves for $Y = 208{\cdot}01$ and $Y = 109{\cdot}20$ also have small imaginary parts. Thus,' the second simple wave for these cases also damps very slowly, though considerably faster than the first. Consequently oscillations of the attenuation factor occur which are due to the interference of the first and second simple waves, and which do not vanish in a large interval of x. The damped oscillations of the function $|\Psi|$ around the almost horizontal asymptote representing the first term of the residue series (4.07) are distinctly seen in Figs. 7 and 8.

We observe that in the case when one of the corresponding points is situated high above the inversion layer (see Chapter 16), the first simple wave which is damped slowly and is therefore responsible for long-range propagation is excited very feebly. Consequently, the first wave determines the field only in a region far inside the shadow where the several succeeding simple waves, which are important for the field near the horizon, are nearly damped out. In the case of intralayer propagation, however, the first simple wave ($Y = 208{\cdot}01$ and $Y = 109{\cdot}20$) not only damps slowly but is excited with a high amplitude.

For the rest of the values of the parameter Y there are no trapped waves. However, in these cases there are also slightly damped simple

waves, and the field decays considerably more slowly than in the absence of inversion. For comparison, we give on Fig. 4, dot-dash curves representing fields obtained in the absence of refraction for $y = y' = 1$ and

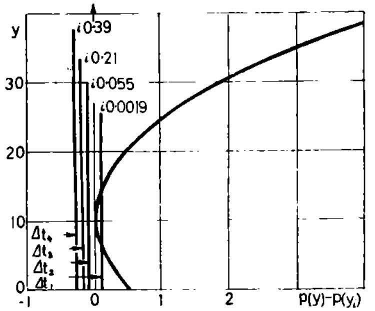

FIG. 5. Roots t_m, corresponding to trapped and untrapped waves (Series I).

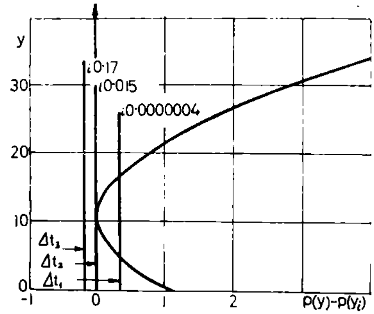

FIG. 6. Roots t_m, corresponding to trapped and untrapped waves (Series II).

$y = y' = 0{\cdot}48$. They run almost vertically and differ sharply from the solid curves for $Y = 52{\cdot}5$ and $Y = 25{\cdot}24$, which give the field at the same height as for the dot–dash curves but for the inversion case.

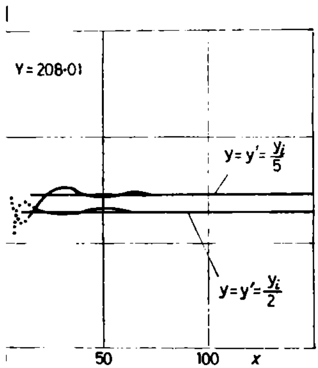

FIG. 7. The function $\Psi(x)$ for $Y = 208{\cdot}01$.

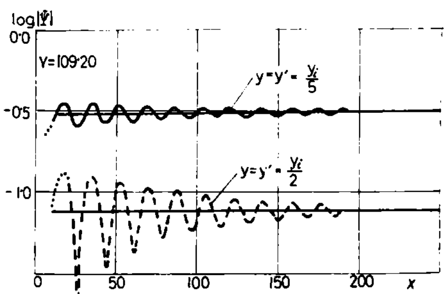

FIG. 8. The function $\Psi(x)$ for $Y = 109{\cdot}20$.

Those sections of the curves on which the residue-series terms yield rather approximate results (which are nevertheless qualitatively correct), are traced by dots. These sections approach very closely the short vertical lines which mark for each curve the geometrical boundary between light and shadow in the absence of refraction.

5. *Numerical Results for a Particular Case*

Numerical results can be obtained for particular cases of propagation for waves of different wavelengths on the basis of the non-dimensional curves analysed in Section 4. For the sake of illustration we choose an

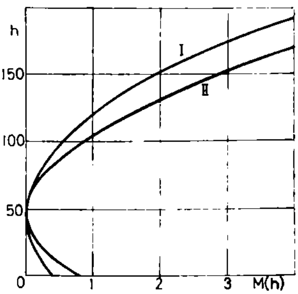

FIG. 9. Dependence of the modified refractive index on height (*M*-profile).

M-profile obtained from the first non-dimensional functions $p(y)$ of the I and II series (Table 3a, line 1 and Table 3b, line 1) by using equation (1.07) (Fig. 9). On Figs. 10 and 11 the corresponding attenuation factors V are given

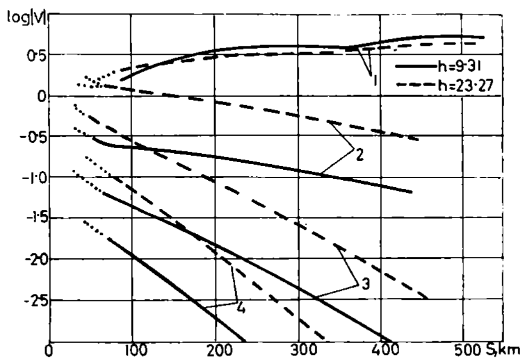

FIG. 10. Dependence of the attenuation factor V on the distance s for wavelengths 1 — 3·33 cm 2 — 10·00 cm 3 — 30·0 cm 4 — 90 cm (*M*-profile I).

for wavelengths of 3·33 cm (curve 1), 10 cm (curve 2), 30 cm (curve 3) and 90 cm (curve 4 on Fig. 10). The height h which is the same for the emission and reception points, is indicated on Figs. 10 and 11 in meters. The distance s is given in kilometers.

The M-profile I has the same inversion height h_i as the M-profile II but an inversion $M(0)-M(h_i)$ which is twice as strong. The attenuation factors for these profiles can be immediately compared since Figs. 10 and 11 correspond to the same height of corresponding points and to the same wavelengths. We thereby get an idea of the influence of the inversion on radiowave propagation (the values of h and λ being prescribed).

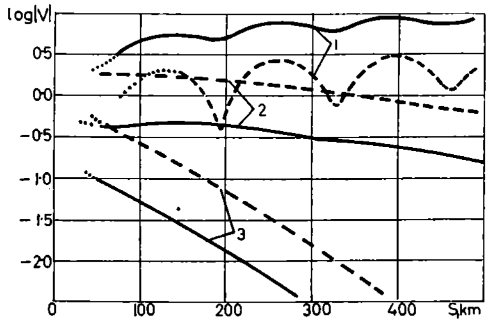

FIG. 11. Dependence of the attenuation factor V on the distance s for wavelengths 1 — 3·33 cm 2 — 10·00 cm 3 — 30·0 cm 4 — 90 cm (M-profile II).
h=9.31 solid curve; h=23.27 dashed curve

The range of propagation for curves 1 and 2 proves to be longer in the layer with the stronger inversion, as would have been expected. However, curves 3 show a rather unexpected effect; these curves do not differ at short ranges but, beginning from a certain distance, the field turns out to be stronger in the case of the weaker inversion. We will consider the physical meaning of this result at the end of Section 6.

6. *Wave Damping in a Tropospheric Waveguide*

As we saw above, the attenuation factor V can be written in form of the residue series (1.22). The complex numbers t_m determine the dependence of the individual terms (or the simple waves) on the dimensionless distance x. They are roots of equation (1.23) or of the more exact equation (2.01). Let us investigate the dependence of the numbers t_m on various factors in more detail.

As has been shown in Section 6 of Chapter 16, equation (1.23) reduces approximately to the simpler equation

$$S_1 = \left(m-\frac{1}{4}\right)\pi \qquad (6.01)$$

Here m is the number of the root $t = t_m$ and S_1 is the integral

$$S_1 = \int_0^{y_1} \sqrt{\{p(y) - t\}}\,\mathrm{d}y \tag{6.02}$$

where y_1 denotes the lesser root of the equation

$$p(y) - t = 0 \tag{6.03}$$

As is easily shown, equations (6.01) and (6.02) remain applicable in the case when the roots of (6.03) are complex if y_1 is understood to be a root with a positive imaginary part. The roots t_m, corresponding in the series (1.22) to both feebly and strongly damped simple waves can be found approximately with the help of these equations. We will make use of equations (6.01) and (6.02) to find out on which parameters the damping of the simple waves depends. We thus get an idea of the dependence of the radiowave propagation range on the parameters of the M-profile with inversion and an estimate of the conditions for long-range propagation.

It seems, at first sight, that the fundamental parameters are the inversion height h_i and the complete increment $M(0) - M(h_i)$ of the modified index in the inversion layer. Indeed, they define the so-called "critical wavelength" λ_m for the mth simple wave in a tropospheric waveguide near the earth

$$\lambda_m = \frac{10^{-3}}{m - \frac{1}{4}} h_i \sqrt{\{2[M(0) - M(h_i)]\}} \tag{6.04}$$

(see Ref. 25).

As a matter of fact, however, the attachment of fundamental importance to these parameters is incorrect. In order to investigate this question, let us introduce instead of y and $p(y)$ the following new reduced variables

$$z = \frac{y}{y_i} = \frac{h}{h_i}; \qquad q(z) = 4\frac{p(y) - p(y_i)}{p(0) - p(y_i)} = 4\frac{M(h) - M(h_i)}{M(0) - M(h_i)} \tag{6.05}$$

so that the function $q(z)$ always satisfies the relations

$$q(0) = 4; \qquad q(1) = 0 \tag{6.06}$$

Instead of the variable t we introduce the quantity

$$\tau = 4\frac{t - p(y)}{p(0) - p(y_i)} \tag{6.07}$$

Then, equation (6.01) becomes

$$\int_0^{z_1} \sqrt{[q(z)-\tau]}\,\mathrm{d}z = \frac{\left(m-\frac{1}{4}\right)\pi}{\sqrt{G}} \tag{6.08}$$

where

$$G = \frac{y_i^2}{4}[p(0)-p(y_i)] = \frac{k^2h_i^2}{2}10^{-6}[M(0)-M(h_i)] \tag{6.09}$$

and z_1 is the root of the equation

$$q(z)-\tau = 0 \tag{6.10}$$

corresponding to y_1.

For a given M-profile with an inversion the critical wavelength λ_m is determined from the relation

$$\int_0^1 \sqrt{[q(z)]}\,\mathrm{d}z = \frac{\left(m-\frac{1}{4}\right)\pi}{\sqrt{G_m}} \tag{6.11}$$

where G_m is the value of G for $\lambda = \lambda_m$. If we use the notation

$$\beta = \int_0^1 \sqrt{[q(z)]}\,\mathrm{d}z \tag{6.12}$$

then for the majority of M-curves the value of β will be almost unity, because of formulae (6.06). Thus, for the M-profiles considered above we have $\beta = 0{\cdot}991$ (series I) and $\beta = 0{\cdot}983$ (series II). We obtain from formula (6.11)

$$\sqrt{G_m} = \frac{\left(m-\frac{1}{4}\right)\pi}{\beta} \tag{6.13}$$

from which we find with help of (6.09)

$$\lambda_m = \frac{\beta\cdot 10^{-3}}{m-\frac{1}{4}} h_i \sqrt{\{2[M(0)-M(h_i)]\}} \tag{6.14}$$

This expression differs from (6.04) only by the factor β.

The designation "critical wavelength" has been introduced in the literature because of the following considerations. For $G > G_m$, i.e., for $\lambda < \lambda_m$ equation (6.08) has a real root $\tau = \tau_m$, within the limits $0 < \tau < 4$. In the series (1.22) this root corresponds to an undamped simple wave.

For $G < G_m$, i.e., for $\lambda > \lambda_m$ equation (6.08) has a complex root, which determines a damped simple wave.

In the following we will use the terminology "critical wavelength", meaning it to be an abbreviated designation for the expression (6.14). However, since no qualitative discontinuity occurs at the critical wavelength, this designation is conditional (see below).

Equation (6.08) can be rewritten in the form

$$\int_0^{z_1} \sqrt{[q(z)-\tau]}\,\mathrm{d}z = \beta\sqrt{\left(\frac{G_m}{G}\right)} = \beta\frac{\lambda}{\lambda_m} \tag{6.15}$$

from which it is seen that for a given M-profile (and, therefore, for a fixed function $q(z)$) all the roots τ must depend only on the ratio λ_m/λ. Instead of this ratio, it is more convenient to use its common logarithm:

$$l = \log\frac{\lambda_m}{\lambda} = \log\sqrt{\left(\frac{G}{G_m}\right)} \tag{6.16}$$

Thus, if the different M-curves have coincident functions $q(z)$, then the corresponding roots τ must fall on the same curve

$$\tau = f(l) \tag{6.17}$$

These conclusions are based on equation (6.01) for the roots t_m. We did not use this equation in our actual calculations since it is too rough, but the values of τ found from the more exact equations are approximately subject to this law. In Fig. 12 are shown the functions $q(z)$ for two series of M-curves which we considered above. For $0 < z < 2$ these functions practically coincide although the M-curves themselves differ noticeably (see Fig. 9). Therefore, the values of τ for all the roots t_m calculated earlier (see Table 4) will lie near a certain common curve (6.17). This is seen from Fig. 13 where we have plotted values of $\mathrm{Im}\tau$, corresponding to the roots t_m in Table 4 (series I) (white circles), values of t_m improved by using the formulae of Section 2 (see Table 1) (black circles) and values from Table 4 (series II) (triangles). The figures near the circles and triangles denote the number of the $p(y)$ curve (first figure) and the number m of the root t_m (second figure). Thus, for example, 3·2 means the second root for the third $p(y)$ curve. Let us recall that the imaginary parts of the quantities τ_m and t_m determine the damping of the simple waves in the series (1.22).

We also see that no qualitative change in the damping of the mth simple wave occurs at $l = 0$, i.e., for $\lambda = \lambda_m$. According to the rigorous theory,

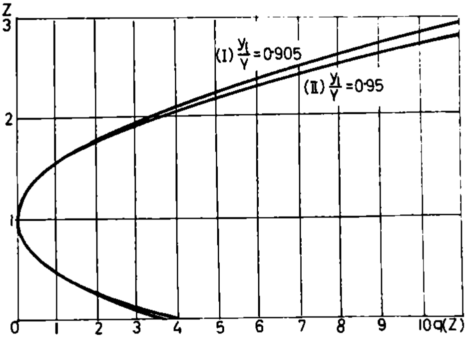

FIG. 12. The function $q(z)$ for the series I and II.

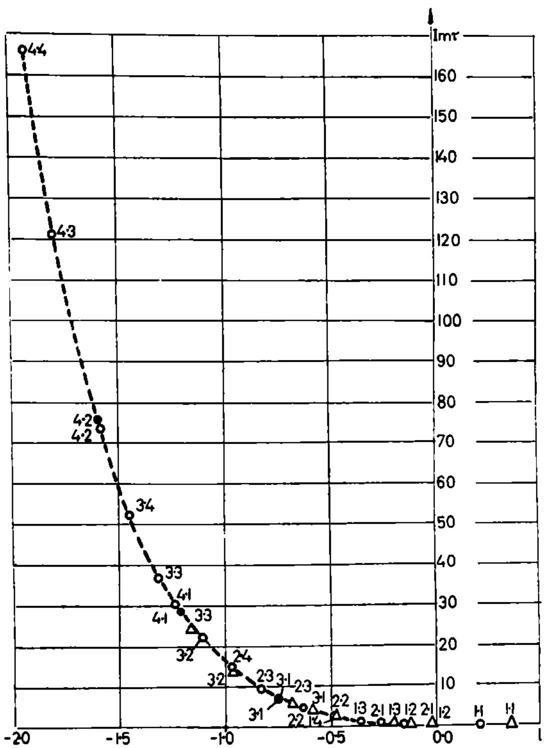

FIG. 13. Imaginary parts of the roots of equation (6.15) as a function of the variable (6·16).

this damping takes place for both $\lambda < \lambda_m$ and $\lambda > \lambda_m$ and increases slowly as λ increases. Consequently, long-range propagation can be observed (though in an attenuated form) for wavelengths λ exceeding the largest critical value λ_1 by a factor of the order 10. In fact even the curve 2 on Fig. 10 corresponds to a wavelength λ larger than λ_1, and long-range propagation occurs even for still longer waves (curves 3 and even 4).

If for the given M-profile the function $q(z)$ differs strongly from the functions represented in Fig. 12, then the behaviour of the curves (6.17) will be qualitatively the same but the numerical relations will be completely different. Thus, assuming

$$q(z) = 4\left(1 - \frac{z - \dfrac{z^n}{n}}{1 - \dfrac{1}{n}}\right) \tag{6.18}$$

and taking $n = 1/2$ and $n = 1/5$, we come to the cases considered in [23] and [24]. These functions $q(z)$ are plotted on Fig. 14. They differ from the functions plotted on Fig. 12 considerably, and consequently the values of

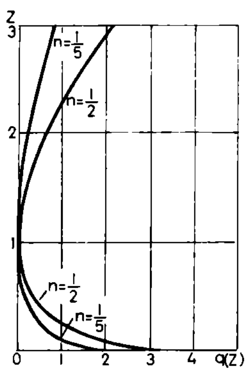

FIG. 14. The functions $q(z)$ for M-profiles, adopted in [23] and [24].

τ for them (for the same l) are approximately an order of 10 less than on Fig. 13. This latter statement follows from Figs. 4,9 of the paper [24] by D. Hartree and others if the relations connecting Hartree's notations with ours are taken into account.

Thus, Fig. 13 is not universal and it is impossible to calculate the damping of simple waves with its aid for arbitrary shapes of the M-profile;

the hypothesis that long-range propagation is determined exclusively by the inversion height h_i and the increment $M(0)-M(h_i)$ proves to be inconsistent. It follows that at least one more parameter exists which characterizes the M-curve and is of primary importance for long-range propagation. It is natural to take as the additional parameter the curvature of the M-curve at the inversion point, i.e., the value $M''(h_i)$, since this parameter substantially influences the leakage of the electromagnetic field from the inversion layer.

If we accept the assumption that the value of $M''(h_i)$ is also a fundamental parameter, then the following new reduced quantities should be introduced in place of $q(z)$ and τ

$$Q(z) = \frac{q(z)}{q''(1)}\,; \qquad T = \frac{\tau}{q''(1)} \tag{6.19}$$

where for a hyperbolic inversion law

$$q''(1) = 8\,\frac{y_1}{Y} \tag{6.20}$$

and for the power law (6.18).

$$q''(1) = 4n \tag{6.21}$$

The curve A on Fig. 15 represents an average line near which the values of τ given on Fig. 13 are grouped. The curve B is a similar average line

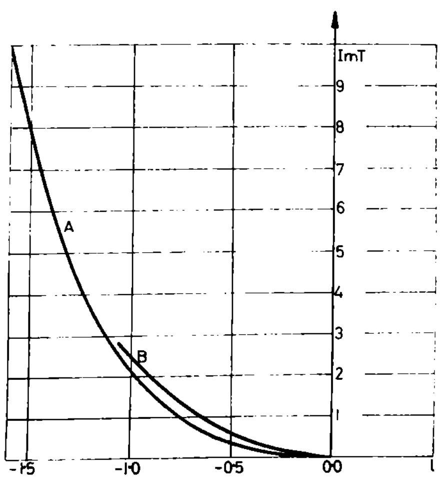

FIG. 15. Comparison of the quantities Im(T) for M-profiles of different type.

constructed with the values plotted on Figs. 4, 9 of Hartree's paper [24] for the function (6.18) with $n=1/5$.

Comparing the curves A and B on Fig. 16, we see that if the reduced quantities (6.19) are used two perfectly different types of M-profiles (see Figs. 12 and 14) yield values of the damping coefficients of simple waves in a tropospheric waveguide adjacent to the earth which are not too different. Consequently, the reduced quantities (6.19) are more convenient for a comparison of the properties of different inversion layers than are the quantities (6.05) and (6.07). This result confirms the hypothesis made above that $M''(h_i)$ is also a fundamental parameter which determines the leakage of electromagnetic energy out of the inversion layer. Let us observe that the real parts of T differ more strongly for the two types of inversion considered.

Since, according to the series (1.22), the decrease in the amplitude of the mth simple wave is determined by the factor

$$e^{-\mathrm{Im}\, t_m x} = e^{-\varkappa_m s} \tag{6.22}$$

where s is the horizontal distance between the corresponding points, then a more descriptive characteristic of the wave damping is given by the coefficient

$$\varkappa_m = \frac{2\pi}{\lambda_m} 10^{-6} h_i^2 M''(h_i)\Theta \tag{6.23}$$

where

$$\Theta = \frac{\lambda_m}{\lambda} \mathrm{Im} T \tag{6.24}$$

The quantity Θ as a function of l is plotted on Fig. 16, where the curves A and B have been constructed from the curves A and B of Fig. 15. Formula (6.23) can be rewritten as follows:

$$\varkappa_m = 2\pi\left(m-\frac{1}{4}\right)10^{-3}\frac{h_i M''(h_i)}{\beta\sqrt{\{2[M(0)-M(h_i)]\}}}\Theta \tag{6.25}$$

Since Θ is the only quantity in the formulae (6.23) and (6.25) depending on wavelength, Fig. 16 directly indicates the character of the dependence of the damping on wavelength. As the wavelength increases the damping at first increases but then it reaches a maximum and begins to decrease. This latter circumstance is easy to understand physically since the propagation of sufficiently long waves is only slightly influenced by the inversion layer. Such radiowaves are propagated farther beyond the horizon the larger their wavelength, as in the case of normal refraction.

The results obtained show that when all three parameters of the M-curve (namely h_i, $(M(0)-M(h_i))$, $M''(h_i)$) are taken into account then the values obtained for the damping coefficients of simple waves are of the correct order of magnitude even for strongly different M-profiles. But even for fixed values of the parameters they may differ somewhat for different M-profiles (as much as 50 per cent or more, see Fig. 16). This means that for a still more exact account of the M-profile further parameters are needed. This question requires additional investigation.

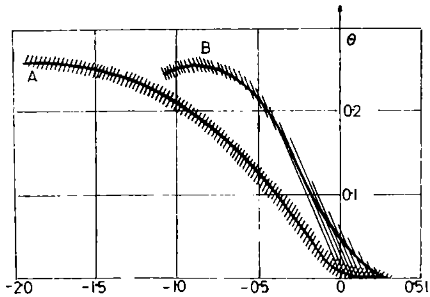

FIG. 16. Damping coefficient of a simple wave propagated along a tropospheric waveguide

Propagation in a waveguide adjacent to the earth can be estimated by comparing the damping coefficient $\varkappa_1$ with the damping of the first simple wave in the theory of radiowave propagation under normal refraction. This comparison can show whether such propagation can be considered as "long-range". This criterion of long-range propagation is somewhat uncertain, but the uncertainty is due to the nature of the phenomenon, which does not undergo discontinuous changes as the wavelength and the inversion layer change continuously.

In the computations for the second series of M-profiles considered above we have taken twice as large a value of $M(0)-M(h_i)$ for the same height of the inversion layer as in the first series, and have thus increased the critical wavelength (6.14). Now, shorter waves are propagated better in waveguides with longer rather than with smaller critical wavelength. On the other hand if the curvature of the M-curve at the inversion point is increased (as in the second series) the leakage of electromagnetic energy of the longer waves from the inversion layer is facilitated. Consequently, long-range propagation of shorter waves will be better in the case of the first series while for longer waves the situation will be reversed (compare curve 3 on Fig. 11 with curve 3 on Fig. 10).

For the calculation of the field in the inversion layer, the residue series (1.22) is most suitable. Physically, this means that medium and long-range radiowave propagation between two corresponding points within an inversion layer can be considered as the transmission of cylindrical waves along a kind of transmission line. The damping of these waves is due to the losses by radiation, their amplitudes R_m depend on the height distribution of the field. The results obtained above illustrate some characteristics of radiowave transmisson along a tropospheric waveguide adjacent to the earth.

Conclusion

In the present paper mathematical methods have been developed, which permit computations of radiowave propagation in an inversion layer near the earth, i.e., in a tropospheric waveguide (Sections 1–3). Using these methods, calculations have been made for a number of particular cases of propagation along a tropospheric waveguide (Section 4). The results obtained allow a comparison of the long-range propagation of radiowaves of different wavelengths (Section 5).

It follows from the results obtained that long-range (waveguide) propagation attentuates very slowly as the wavelength increases. Thus, for example, the wavelength can exceed the so-called critical wavelength λ_m by a factor of 10 and long-range propagation will still take place. The criterion for long-range propagation has been made more precise (Section 6).

APPENDIX

Tables of Airy Functions

In our investigations on the theory of electromagnetic diffraction and radiowave propagation, extensive use is made of Airy functions. It is therefore appropriate to include tables of Airy functions in this Collection, together with a brief account of their properties and possible applications.

1. *Definition and Fundamental Properties of Airy Functions*

The name Airy functions will be attached to functions connected with the well-known Airy integral

$$v(t) = \frac{1}{\sqrt{\pi}} \int_0^\infty \cos\left(\frac{x^3}{3} + xt\right) \mathrm{d}x \tag{1.01}$$

first introduced in 1838 in Airy's investigations "On the intensity of light in the neighbourhood of a caustic" *(Trans. Camb. Phil. Soc.* **VI** (1838) p. 379).

The Airy integral represents one of the solutions of the differential equation

$$w''(t) = tw(t) \tag{1.02}$$

namely, that solution which tends to zero faster than any finite power of t when t tends to positive infinity. Together with this solution, $v(t)$, we will consider the other, linearly independent, solution $u(t)$, to be defined later. The functions $u(t)$ and $v(t)$ will be called Airy functions.

Let us consider the integral

$$w(t) = \frac{1}{\sqrt{\pi}} \int_\Gamma e^{tz - \frac{1}{3}z^3} \mathrm{d}z \tag{1.03}$$

where the integration path Γ in the complex z-plane runs from infinity to zero along the line arc $z = -2\pi/3$ and from zero to infinity along the line arc $z=0$ (along the positive real axis).

The integral (1.03) converges for all complex values of t and represents an integral transcendental function of t. It is easy to verify that the func-

tion $w(t)$ so defined satisfies the differential equation (1.02). The values of the function $w(t)$ and of its derivative $w'(t)$ at $t=0$ are

$$w(0) = \frac{2\sqrt{\pi}}{3^{2/3}\Gamma\left(\frac{2}{3}\right)} e^{i\frac{\pi}{6}} = 1{\cdot}0899290710+i0{\cdot}6292708425 \tag{1.04}$$

$$w'(0) = \frac{2\sqrt{\pi}}{3^{4/3}\Gamma\left(\frac{4}{3}\right)} e^{-i\frac{\pi}{6}} = 0{\cdot}7945704238-i0{\cdot}4587454481 \tag{1.05}$$

The function $w(t)$, being an integral transcendental function, admits an expansion in a power series which is convergent for all values of t. This series is of the form

$$\begin{aligned} w(t) = {} & w(0)\left[1+\frac{t^3}{2\cdot 3}+\frac{t^6}{(2\cdot 5)\cdot(3\cdot 6)}+\frac{t^9}{(2\cdot 5\cdot 8)\cdot(3\cdot 6\cdot 9)}+\ldots\right] \\ & +w'(0)\left[t+\frac{t^4}{3\cdot 4}+\frac{t^7}{(3\cdot 6)\cdot(4\cdot 7)}+\frac{t^{10}}{(3\cdot 6\cdot 9)\cdot(4\cdot 7\cdot 10)}+\ldots\right] \end{aligned} \tag{1.06}$$

Taking t real, we separate in $w(t)$ the real and the imaginary part putting

$$w(t) = u(t)+iv(t) \tag{1.07}$$

The functions $u(t)$ and $v(t)$ so defined will be two independent solutions of the differential equation (1.02) connected by the relation

$$u'(t)v(t)-u(t)v'(t) = 1 \tag{1.08}$$

The function $v(t)$ defined according to (1.07) will be the same as that defined by means of the Airy integral (1.01). We are therefore entitled to call both functions $u(t)$ and $v(t)$ Airy functions.

The Airy functions are real for real values of t, but since they are integral transcendental functions they are defined also for all complex values of t. In the complex t-plane the following relations hold

$$w(t) = u(t)+iv(t) \tag{1.09}$$

$$w\left(te^{i\frac{\pi}{3}}\right) = 2e^{i\frac{\pi}{6}}v(-t) \tag{1.10}$$

$$w\left(te^{i\frac{2\pi}{3}}\right) = e^{i\frac{\pi}{3}}[u(t)-iv(t)] \tag{1.11}$$

$$w(te^{i\pi}) = u(-t)+iv(-t) \tag{1.12}$$

$$w\left(te^{i\frac{4\pi}{3}}\right) = 2e^{i\frac{\pi}{6}}v(t) \tag{1.13}$$

$$w\left(te^{i\frac{5\pi}{8}}\right) = e^{i\frac{\pi}{3}}[u(-t)-iv(-t)] \tag{1.14}$$

By means of these relations the values of the function $w(t)$ on the six rays arc $t = n\pi/3$ ($n = 0, 1, 2, 3, 4, 5$) in the complex t-plane are expressed in terms of the real Airy functions $u(t)$ and $v(t)$.

2. *Asymptotic Expressions for the Airy Functions*

We suppose that t is large and positive and put

$$x = \frac{2}{3} t^{3/2} \tag{2.01}$$

We further denote by the symbol $F_{20}(\alpha, \beta, z)$ a formal series of the form

$$F_{20}(\alpha, \beta, z) = 1 + \frac{\alpha \cdot \beta}{1} z + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2} z^2 + \dots \tag{2.02}$$

Then the following asymptotic expressions hold for the Airy functions and their derivatives

$$u(t) = t^{-\frac{1}{4}} e^{x} F_{20}\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2x}\right) \tag{2.03}$$

$$u'(t) = t^{\frac{1}{4}} e^{x} F_{20}\left(-\frac{1}{6}, \frac{7}{6}, \frac{1}{2x}\right) \tag{2.04}$$

$$v(t) = \frac{1}{2} t^{-\frac{1}{4}} e^{-x} F_{20}\left(\frac{1}{6}, \frac{5}{6}, -\frac{1}{2x}\right) \tag{2.05}$$

$$v'(t) = -\frac{1}{2} t^{\frac{1}{4}} e^{-x} F_{20}\left(-\frac{1}{6}, \frac{7}{6}, -\frac{1}{2x}\right) \tag{2.06}$$

For negative values of the argument the asymptotic expressions of the Airy functions can be obtained by separating the real and the imaginary parts in the formulae

$$w(-t) = t^{-\frac{1}{4}} e^{i\left(x+\frac{\pi}{4}\right)} F_{20}\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2ix}\right) \tag{2.07}$$

$$w'(-t) = t^{\frac{1}{4}} e^{i\left(x-\frac{\pi}{4}\right)} F_{20}\left(-\frac{1}{6}, \frac{7}{6}, \frac{1}{2ix}\right) \tag{2.08}$$

The above expressions are valid not only for real positive values of t, but also in a certain sector including the positive real axis. For different functions this sector is different, but in any case all the above expressions are valid in the sector

$$-\frac{\pi}{3} < \text{arc}\, t < \frac{\pi}{3}$$

If we put

$$F_{20}\left(\frac{1}{6}, \frac{5}{6}, \frac{1}{2x}\right) = 1+\frac{a_1}{x}+\frac{a_2}{x^2}+\frac{a_3}{x^3}+\dots \tag{2.09}$$

then the coefficients $a_1, a_2, \dots$ will be equal to

$$\left.\begin{aligned} a_1 &= \frac{5}{72}; \quad a_2 = \frac{(5\cdot 11)\cdot 7}{1\cdot 2\cdot (72)^2}; \quad a_3 = \frac{(5\cdot 11\cdot 17)(7\cdot 13)}{1\cdot 2\cdot 3\cdot (72)^3} \\ a_n &= \frac{5\cdot 11 \dots (6n-1)\cdot 7\cdot 13 \dots (6n-5)}{1\cdot 2 \dots n(72)^n} \end{aligned}\right\} \tag{2.10}$$

Similarly, in the series

$$F_{20}\left(-\frac{1}{6}, \frac{7}{6}, \frac{1}{2x}\right) = 1-\frac{b_1}{x}-\frac{b_2}{x^2}-\frac{b_3}{x^3}-\dots \tag{2.11}$$

the coefficients are equal to

$$\begin{aligned} b_1 &= \frac{7}{72}; \quad b_2 = \frac{(7\cdot 13)\cdot 5}{1\cdot 2\cdot (72)^2}; \quad b_3 = \frac{(7\cdot 13\cdot 19)\cdot (5\cdot 11)}{1\cdot 2\cdot 3\cdot (72)^3} \\ b_n &= \frac{7\cdot 13 \dots (6n+1)\cdot 5\cdot 11 \dots (6n-7)}{1\cdot 2 \dots n\,(72)^n} \end{aligned} \tag{2.12}$$

The explicit form of the asymptotic expressions for the Airy functions of positive argument is

$$u(t) = t^{-1/4}\, e^{x}\left(1+\frac{a_1}{x}+\frac{a_2}{x^2}+\dots\right) \tag{2.13}$$

$$u'(t) = t^{1/4}\, e^{x}\left(1-\frac{b_1}{x}-\frac{b_2}{x^2}-\dots\right) \tag{2.14}$$

$$v(t) = \frac{1}{2}\, t^{-1/4}\, e^{-x}\left(1-\frac{a_1}{x}+\frac{a_2}{x^2}-\frac{a_3}{x^3}+\dots\right) \tag{2.15}$$

$$v'(t) = -\frac{1}{2}\, t^{1/4}\, e^{-x}\left(1+\frac{b_1}{x}-\frac{b_2}{x^2}+\frac{b_3}{x^3}-\dots\right) \tag{2.16}$$

The corresponding expressions for the Airy functions of negative argument are

$$\begin{aligned} u(-t) = {} & t^{-1/4}\cos\left(x+\frac{\pi}{4}\right)\left[1-\frac{a_2}{x^2}+\frac{a_4}{x^4}-\frac{a_6}{x^6}+\dots\right]+ \\ & +t^{-1/4}\sin\left(x+\frac{\pi}{4}\right)\left[\frac{a_1}{x}-\frac{a_3}{x^3}+\frac{a_5}{x^5}-\frac{a_7}{x^7}+\dots\right] \end{aligned} \tag{2.17}$$

$$u'(-t) = t^{1/4}\sin\left(x+\frac{\pi}{4}\right)\left[1+\frac{b_2}{x^2}-\frac{b_4}{x^4}+\frac{b_6}{x^6}-\dots\right]+$$
$$+t^{1/4}\cos\left(x+\frac{\pi}{4}\right)\left[\frac{b_1}{x}-\frac{b_3}{x^3}+\frac{b_5}{x^5}-\frac{b_7}{x^7}+\dots\right] \qquad (2.18)$$

$$v(-t) = t^{-1/4}\sin\left(x+\frac{\pi}{4}\right)\left[1-\frac{a_2}{x^2}+\frac{a_4}{x^4}-\frac{a_6}{x^6}+\dots\right]-$$
$$-t^{-1/4}\cos\left(x+\frac{\pi}{4}\right)\left[\frac{a_1}{x}-\frac{a_3}{x^3}+\frac{a_5}{x^5}-\frac{a_7}{x^7}+\dots\right] \qquad (2.19)$$

$$v'(-t) = -t^{1/4}\cos\left(x+\frac{\pi}{4}\right)\left[1+\frac{b_2}{x^2}-\frac{b_4}{x^4}+\frac{b_6}{x^6}+\dots\right]+$$
$$+t^{1/4}\sin\left(x+\frac{\pi}{4}\right)\left[\frac{b_1}{x}-\frac{b_3}{x^3}+\frac{b_5}{x^5}-\frac{b_7}{x^7}+\dots\right]. \qquad (2.20)$$

3. *Connection Between the Airy and the BesselFunctions*

The Airy functions of positive argument can be expressed in terms of the Bessel functions of the first and the second kind of order one-third and with an imaginary argument. The Airy functions of negative argument can be expressed in terms of Bessel functions of the first and the second kind of order one-third and with a real argument. Finally, the complex Airy function w is simply expressed in terms of the first Hankel function of order one-third. The derivatives of the Airy functions are expressible in terms of the corresponding Bessel and Hankel functions of order two-thirds.

The notations for the Bessel and Hankel functions will be the same as in Watson's book [18]. Assuming $t > 0$ and putting $x = 2/3 \cdot t^{3/2}$ we have

$$u(t) = \sqrt{\left(\frac{\pi}{3}t\right)}\left[I_{-1/3}(x)+I_{1/3}(x)\right] = \sqrt{\left(\frac{\pi}{3}t\right)}\left[2I_{1/3}(x)+\frac{\sqrt{3}}{\pi}K_{1/3}(x)\right] \qquad (3.01)$$

$$u(-t) = \sqrt{\left(\frac{\pi}{3}t\right)}\left[J_{-1/3}(x)-J_{1/3}(x)\right] = -\sqrt{\left(\frac{\pi}{3}t\right)}\left[\frac{1}{2}J_{1/3}(x)+\frac{\sqrt{3}}{2}Y_{1/3}(x)\right] \qquad (3.02)$$

$$u'(t) = \sqrt{\left(\frac{\pi}{3}\right)}\,t\left[I_{-2/3}(x)+I_{2/3}(x)\right] = \sqrt{\left(\frac{\pi}{3}\right)}\,t\left[2I_{3/2}(x)+\frac{\sqrt{3}}{\pi}K_{2/3}(x)\right] \qquad (3.03)$$

$$u'(-t) = \sqrt{\left(\frac{\pi}{3}\right)}\,t\left[J_{-2/3}(x)+J_{2/3}(x)\right] = \sqrt{\left(\frac{\pi}{3}\right)}\,t\left[\frac{1}{2}J_{2/3}(x)-\frac{\sqrt{3}}{2}Y_{2/3}(x)\right] \qquad (3.04)$$

$$v(t) = \frac{1}{3}\sqrt{(\pi t)}\left[I_{-1/3}(x) - I_{1/3}(x)\right] = \frac{1}{\sqrt{(3\pi)}}\sqrt{t}\cdot K_{1/3}(x) \tag{3.05}$$

$$v(-t) = \frac{1}{3}\sqrt{(\pi t)}\left[J_{-1/3}(x) + J_{1/3}(x)\right] = \sqrt{\left(\frac{\pi}{3}t\right)}\left[\frac{\sqrt{3}}{2}J_{1/3}(x) - \frac{1}{2}Y_{1/3}(x)\right] \tag{3.06}$$

$$v'(t) = -\frac{1}{3}\sqrt{\pi}\cdot t\left[I_{-2/3}(x) - I_{2/3}(x)\right] = -\frac{1}{\sqrt{(3\pi)}}tK_{2/3}(x) \tag{3.07}$$

$$v'(-t) = -\frac{1}{3}\sqrt{\pi}\cdot t\left[J_{-2/3}(x) - J_{2/3}(x)\right] = \sqrt{\left(\frac{\pi}{3}\right)}t\left[\frac{\sqrt{3}}{2}J_{2/3}(x) + \frac{1}{2}Y_{2/3}(x)\right] \tag{3.08}$$

$$w(-t) = \sqrt{\left(\frac{\pi}{3}\right)}e^{i\frac{2}{3}\pi}\sqrt{t}\cdot H^{(1)}_{1/3}(x) \tag{3.09}$$

$$w'(-t) = \sqrt{\left(\frac{\pi}{3}\right)}e^{i\frac{\pi}{3}}tH^{(1)}_{2/3}(x) \tag{3.10}$$

It is important to note that the Airy functions considered as functions of t are integral transcendental functions, while the Bessel and Hankel functions involved in the above formulae are not integral functions of x, but have a singularity at $x=0$. This difference becomes apparent, for small values of the argument, in that the tables of the Airy functions show a more smooth variation as compared with the tables of the Bessel and Hankel function; this circumstance facilitates interpolation considerably. The integral character of the Airy functions is also very useful in theoretical investigations.

4. *Roots of the Airy Functions*

Most important in the applications are the roots of the function $v(t)$ and of its derivative $v'(t)$. Since the Airy functions become oscillatory for negative values of t, these roots are real and negative. The roots of $v(t)$ will be denoted by $-\tau_s^0$ and the roots of $v'(t)$ by $-\tau_s'$, where τ_s^0 and τ_s' are positive quantities. The values of the first five roots and of their common logarithms are given in the following table:

s	τ_s^0	$\log \tau_s^0$	τ_s'	$\log \tau_s'$
1	2·33811	0·368864	1·01879	0·008086
2	4·08795	0·611506	3·24820	0·511642
3	5·52056	0·741983	4·82010	0·683056
4	6·78671	0·831659	6·16331	0·789814
5	7·94417	0·900048	7·37218	0·867596

The subsequent roots can be calculated with help of the formulae for the roots of the Bessel and Neumann functions and their linear combinations (see Watson's book [18]). We have

$$\tau_s^0 = \left(\frac{3}{2} x_s^0\right)^{2/3}; \qquad \tau_s' = \left(\frac{3}{2} x_s'\right)^{2/3} \tag{4.01}$$

where x_s^0 and x_s' satisfy the equations

$$\frac{\sqrt{3}}{2} \cdot J_{1/3}(x_s^0) - \frac{1}{2} Y_{1/3}(x_s^0) = 0 \tag{4.02}$$

$$\frac{\sqrt{3}}{2} J_{2/3}(x_s') + \frac{1}{2} Y_{2/3}(x_s') = 0 \tag{4.03}$$

The following approximate expressions for the quantities x_s^0 and x_s' are known

$$x_s^0 = \left(s - \frac{1}{4}\right)\pi + \frac{0{\cdot}0884194}{4s-1} - \frac{0{\cdot}08328}{(4s-1)^3} + \frac{0{\cdot}4065}{(4s-1)^5} \tag{4.04}$$

$$x_s' = \left(s - \frac{3}{4}\right)\pi - \frac{0{\cdot}1237872}{4s-3} + \frac{0{\cdot}07758}{(4s-3)^3} - \frac{0{\cdot}389}{(4s-3)^5} \tag{4.05}$$

These formulae give very accurate results even for values of s that are rather small. Using them, we easily obtain with help of the relation (4.01) the quantities τ_s^0 and τ_s'.

Similar formulae can be applied to the roots $\tau = \tau_s^0(\sigma)$ of the equation

$$v(-\tau)\cos \pi\sigma - u(-\tau)\sin \pi\sigma = 0 \tag{4.06}$$

and also to the roots $\tau = \tau_s'(\sigma)$ of the equation

$$v'(-\tau)\cos \pi\sigma - u'(-\tau)\sin \pi\sigma = 0 \tag{4.07}$$

For values of s that are not too small the quantities $\tau_s^0(\sigma)$ and $\tau_s'(\sigma)$ can be obtained from the preceding formulae by replacing s by $s+\sigma$ so that we can write formally

$$\tau_s^0(\sigma) = \tau_{s+\sigma}^0; \qquad \tau_s'(\sigma) = \tau_{s+\sigma}' \tag{4.08}$$

In particular, for $\sigma = 1/2$, we have the roots of $u(-\tau)$ and of $u'(-\tau)$.

The roots t_s^0 of the complex Airy function $w(t)$ as well as the roots t_s' of its derivative $w'(t)$ are situated on the ray arc $t = \pi/3$ and can be expressed in terms of the roots τ_s^0 and τ_s' of the functions $v(-\tau)$ and $v'(-\tau)$ by means of the formulae

$$t_s^0 = \tau_s^0 e^{i\frac{\pi}{3}}; \qquad t_s' = \tau_s' e^{i\frac{\pi}{3}} \tag{4.09}$$

Thus the table above gives at the same time the moduli of the complex quantities t_s^0 and t_s'.

5. *Application of the Airy Functions to the Asymptotic Integration of Linear Differential Equations of Second Order*

The Airy functions have various applications in mathematical physics and, in particular, in the theory of diffraction. The mathematical basis of most of these applications is the asymptotic integration of the differential equation

$$\frac{d^2y}{dx^2} = k^2p(x)y \tag{5.01}$$

involving a large parameter k. If the function $p(x)$ does not change sign and satisfies certain continuity conditions in some given interval of the variable x, then the solution of the equation (5.01) can be approximately expressed in terms of exponential or circular functions.

If $p(x)$ is positive, we have either

$$y \cong \frac{C_1}{\sqrt[4]{\{p(x)\}}} \exp\left(k \int_{x_0}^{x} \sqrt{\{p(x)\}}\, dx\right) \tag{5.02}$$

or

$$y \cong \frac{C_2}{\sqrt[4]{\{p(x)\}}} \exp\left(-k \int_{x_0}^{x} \sqrt{\{p(x)\}}\, dx\right) \tag{5.03}$$

according to the condition imposed on the solution (whether it should be an increasing or a decreasing function of $x-x_0$). There is, in general, no sense in taking the sum of the two expressions (5.02) and (5.03), since in the case when the constants C_1 and C_2 are of the same order the decreasing integral will be small not only compared with the whole value of the increasing one, but also compared with the error involved in the asymptotic expression for the latter.

If $p(x)$ is negative, the asymptotic expression for y is of the form

$$y = \frac{C_1'}{\sqrt[4]{[-p(x)]}} \cos\left(k \int_{x_0}^{x} \sqrt{[-p(x)]}\, dx\right) + \\ + \frac{C_2'}{\sqrt[4]{[-p(x)]}} \sin\left(k \int_{x_0}^{x} \sqrt{[-p(x)]}\, dx\right) \tag{5.04}$$

Expressions (5.02)–(5.04) cease to be applicable, if the function $p(x)$ becomes zero within the range of variation of the variable x considered. But, if the root of $p(x)$ is simple, the solution of equation (5.01) can be approximately expressed in terms of Airy functions.

We make a change of variables in the equation (5.01) by introducing a new independent variable ζ and a new function z according to the formulae

$$x = x(\zeta); \qquad y(x) = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)}\, z(\zeta) \tag{5.05}$$

If y satisfies equation (5.01), then the differential equation for z will be of the form

$$\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} = \left[s(\zeta) + k^2 p(x)\left(\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)^2\right] z \tag{5.06}$$

where the symbol $s(\zeta)$ denotes the differential expression

$$s(\zeta) = -\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}\left(\log\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right) + \frac{1}{4}\left[\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\log\frac{\mathrm{d}x}{\mathrm{d}\zeta}\right)\right]^2 \tag{5.07}$$

usually called the Schwartzian derivative.

Let $x = x_0$ be a simple root of the function $p(x)$ so that

$$p(x_0) = 0; \qquad p'(x_0) \neq 0$$

For the sake of definiteness we suppose $p'(x_0) > 0$ and make the substitution

$$k\int_{x_0}^{x} \sqrt{[p(x)]}\,\mathrm{d}x = \frac{2}{3}t^{3/2} \qquad (x > x_0,\ t > 0) \tag{5.08}$$

$$k\int_{x}^{x_0} \sqrt{[-p(x)]}\,\mathrm{d}x = \frac{2}{3}(-t)^{3/2} \qquad (x < x_0,\ t < 0) \tag{5.09}$$

In the vicinity of $x = x_0$ both formulae yield

$$t = k^{2/3}[p'(x_0)]^{1/3}(x - x_0) + \dots \tag{5.10}$$

so that t is a holomorphic function of $x - x_0$ and, inversely, x is a holomorphic function of t:

$$x = x_0 + k^{-2/3}[p'(x_0)]^{-1/3}t + \dots \tag{5.11}$$

To values of $x - x_0$ of the order unity correspond values of t of the order $k^{2/3}$.

As a consequence of the formulae (5.08) – (5.09) we have

$$k^2 p(x)\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 = t \tag{5.12}$$

If in the differential equation (5.06) we put $\zeta = t$ for z, we obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = (s + t)z \tag{5.13}$$

An estimate of the function $s(t)$ shows that in general this function will be very small (of the order $k^{-4/3}$) for small as well as for large values of t (if t is near zero and if t is large of the order $k^{2/3}$). The function $s(t)$ becomes large only if x is near to the next root $x=x_1$ of the function $p(x)$. Excluding this case, we can neglect in the equation (5.13) the term s in the coefficient of z, and we thus obtain

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = tz \tag{5.14}$$

i.e. the differential equation of the Airy functions.

We write the general solution of equation (5.14) in the form

$$z = Au(t)+Bv(t) \tag{5.15}$$

where A and B are constants. Returning to the original function $y(x)$, we will have

$$y = \frac{1}{\sqrt{k}} \sqrt[4]{\left(\frac{t}{p(x)}\right)} \left[Au(t)+Bv(t)\right] \tag{5.16}$$

where t is defined by (5.08) and (5.09).

We now establish the connection between this expression for y and the previous expressions (5.02)—(5.04).

When the difference $x-x_0$ is finite and positive, the quantity t will be positive and very large. For such values of t we have to distinguish two cases: $A\neq 0$ and $A=0$. In the first case, when A is different from zero, the solution is an increasing function, whose asymptotic expression will be obtained from (5.16) by neglecting the term with $v(t)$ and by replacing $u(t)$ by the function

$$u(t) = t^{-1/4} \exp\left(\frac{2}{3} t^{3/2}\right) \tag{5.17}$$

which represents the principal term of the asymptotic expansion considered in Section 2. Using (5.08), we obtain

$$y = \frac{A}{\sqrt{k}\cdot\sqrt[4]{\{p(x)\}}} \exp\left(k\int_{x_0}^{x} \sqrt{\{p(x)\}}\,\mathrm{d}x\right) \tag{5.18}$$

The expression is of the form (5.02) with the value $C_1=A/\sqrt{k}$ for the constant C_1. The constant B does not enter in this expression, and therefore one and the same asymptotic expression can belong to different solutions.

In the second case, when the constant A is equal to zero, only the second term remains in (5.16). Replacing the function $v(t)$ in (5.16) by the quantity

$$v(t) = \frac{1}{2} t^{-1/4} \exp\left(-\frac{2}{3} t^{3/2}\right) \tag{5.19}$$

we obtain

$$y = \frac{B}{2\sqrt{k}\cdot\sqrt[4]{[p(x)]}} \exp\left(-k\int_{x_0}^{x} \sqrt{[p(x)]}\,\mathrm{d}x\right); \qquad A = 0 \tag{5.20}$$

In this case the solution is a decreasing function, asymptotically equal to (5.03) with the value $B/(2\sqrt{k})$ of the constant C_2.

We now consider finite and negative values of $x-x_0$ to which correspond large and negative values of t. In this case we need not distinguish the two cases ($A\neq 0$ and $A=0$), because both Airy functions will be of the same order. Using the approximate expressions

$$u(t) = (-t)^{-1/4} \cos\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] \tag{5.21}$$

$$v(t) = (-t)^{-1/4} \sin\left[\frac{2}{3}(-t)^{3/2} + \frac{\pi}{4}\right] \tag{5.22}$$

for these functions, we obtain, with help of (5.09), the following formula for y:

$$y = \frac{A}{\sqrt{k}\cdot\sqrt[4]{[-p(x)]}} \cos\left(k\int_{x}^{x_0} \sqrt{[-p(x)]}\,\mathrm{d}x + \frac{\pi}{4}\right) + \\ + \frac{B}{\sqrt{k}\cdot\sqrt[4]{[-p(x)]}} \sin\left(k\int_{x}^{x_0} \sqrt{[-p(x)]}\,\mathrm{d}x + \frac{\pi}{4}\right) \tag{5.23}$$

This expression is easily reduced to the form (5.04), with the values

$$C_1' = \frac{A+B}{\sqrt{(2k)}}; \qquad C_2' = \frac{A-B}{\sqrt{(2k)}} \tag{5.24}$$

of the constants C_1' and C_2'.

The expression in terms of Airy functions obtained above for the solution of the differential equation (5.01) thus reduces in limiting cases to simpler expressions in terms of exponential and circular functions. The expression in terms of Airy functions has the essential advantage that it applies uniformly to the whole interval, including the vicinity of the root $x=x_0$ of the function $p(x)$, while the expressions in terms of elementary functions are applicable only sufficiently far from the root. As to the

practical convenience of our expression for numerical computations, the use of Airy functions, once they are tabulated, is in no respect more difficult than the use of elementary functions.

Many functions encountered in mathematical physics satisfy differential equations of the form (5.01) or are reducible to functions satisfying this equation. Therefore, the approximate expressions for these functions given above can have wide applications. In quantum mechanics similar expressions have been proposed by Kramers in connection with the rule of "half-integral" quantization. In the theory or Bessel functions the results obtained here can be used to deduce asymptotic formulae valid in the case when the order and the argument of the Bessel function are large and nearly equal. Such expressions were developed in Ref. 19.

6. *Application of the Airy Functions to the Asymptotic Representation of the Hankel Functions*

As a second typical example of applications of the Airy functions we shall consider the deduction of an asymptotic expression for the Hankel function $H_\nu^{(1)}(\varrho)$ whose order ν and argument ϱ are large and nearly equal, in the sense that the ratio

$$\frac{\nu-\varrho}{\sqrt[3]{\left(\frac{\varrho}{2}\right)}} = t \tag{6.01}$$

remains finite while ν and ϱ are very large. This deduction is based not on the differential equation for the Airy functions, but on their integral representation (1.03).

The Hankel function $H_\nu^{(1)}(\varrho)$ admits the integral representation

$$H_\nu^{(1)}(\varrho) = \frac{1}{\pi i}\int_C \mathrm{e}^{-\varrho \sinh v+\nu v}\,\mathrm{d}v \tag{6.02}$$

where the contour C runs along the straight line $\mathrm{Im}(v) = -\pi$ from $-\pi i-\infty$ to some point $v=v_0$ in the third quadrant of the complex v—plane (e.g. the point $v_0 = -\pi/\sqrt{3}-i\pi$), then along the straight line joining this point $v=v_0$ with the origin $v=0$ and then from zero to infinity along the real axis. We express, according to (6.01), ν in terms of t and introduce the quantity

$$z = v\cdot \sqrt[3]{\left(\frac{\varrho}{2}\right)} \tag{6.03}$$

as integration variable.

The integrand in (6.02) can be written as a product of two factors: the factor

$$\exp\left[(\nu-\varrho)v-\frac{\varrho v^3}{6}\right] = \exp\left(tz-\frac{z^3}{3}\right) \tag{6.04}$$

not involving ϱ explicitly and the factor

$$\exp\left[-\varrho\sinh v+\varrho v+\varrho\frac{v^3}{6}\right] = 1-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3} z^5\ldots \tag{6.05}$$

which, for finite z and large ϱ, can be expanded in fractional negative powers of ϱ (multiples of $-2/3$). Inserting these expressions in the integral (6.02), we obtain

$$H_\nu^{(1)}(\varrho) = \frac{1}{\pi i}\left(\frac{\varrho}{2}\right)^{-1/3}\int_\Gamma e^{tz-\frac{1}{3}z^3}\left[1-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3} z^5+\ldots\right]dz \tag{6.06}$$

where the contour Γ in the z-plane corresponds to the contour C in the v-plane. The relevant part of this contour Γ is identical with that of the contour Γ in formula (1.03). Evaluating the integrals in (6.06) with help of (1.03), we obtain

$$H_\nu^{(1)}(\varrho) = -\frac{i}{\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^{-1/3}\left\{w(t)-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3} w^{(5)}(t)+\ldots\right\} \tag{6.07}$$

By virtue of the differential equation (1.02) the fifth derivative is equal to

$$w^{(5)}(t) = t^2w'(t)+4tw(t) \tag{6.08}$$

Inserting this into (6.07) we obtain the required asymptotic expression for the Hankel function

$$H_\nu^{(1)}(\varrho) = -\frac{i}{\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^{-1/3}\left\{w(t)-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}[t^2w'(t)+4tw(t)]+\ldots\right\} \tag{6.09}$$

Separating the real and the imaginary parts, we arrive to the following asymptotic expressions for the Bessel and Neumann functions in terms of the tabulated Airy functions

$$J_\nu(\varrho) = \frac{1}{\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^{-1/3}\left\{v(t)-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}[t^2v'(t)+4tv(t)]+\ldots\right\} \tag{6.10}$$

$$Y_\nu(\varrho) = -\frac{1}{\sqrt{\pi}}\left(\frac{\varrho}{2}\right)^{-1/3}\left\{u(t)-\frac{1}{60}\left(\frac{\varrho}{2}\right)^{-2/3}[t^2u'(t)+4tu(t)]+\ldots\right\} \tag{6.11}$$

The expressions obtained are valid for complex values of t as well and can be used, in particular, for the approximate determination of the roots of the Hankel function considered as a function of ν.

7. *Explanation of the Tables*

In the following we give tables of the Airy functions $u(t)$, $v(t)$ and of their derivatives $u'(t)$, $v'(t)$. Originally, these tables were calculated with a greater number of decimals, but the results were rounded up to four decimals. For negative values of t, for which the Airy functions are of oscillating character, the tables are given to four places of decimals. For positive values of t, for which the Airy functions are monotonic, the tables are given to four significant figures if the first figure is 2, 3, 4, 5, 6, 7, 8, 9, and to five significant figures if the first figure is 1. The range of the argument t is from $-$ 9·00 to +9·00 and the interval is 0·02. Such a small tabular interval is chosen in order to facilitate interpolation. In most cases linear interpolation is sufficient; in exceptional cases second order interpolation may become necessary. Since together with the values of the functions their first differences are also given, the interpolation of our tables is an easy matter.

For positive values of the argument the functions $u(t)$ and $u'(t)$ rapidly increase, while the functions $v(t)$ and $v'(t)$ rapidly decrease. Therefore, in some ranges the values of the functions $u(t)$ and $u'(t)$ are divided by 10^3 and by 10^6, and the values of the functions $v(t)$ and $v'(t)$ are multiplied by 10^3, 10^6, or 10^9.

For values of the argument outside the range ($-9{\cdot}00$, $+9{\cdot}00$) the Airy functions and their derivatives are easily calculated from the asymptotic expressions given in Section 2; in these expressions it is sufficient to take two or three terms at most.

TABLES OF THE FUNCTIONS $u(t)$, $v(t)$ AND THEIR DERIVATIVES

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−9·00	0·5760	−31	−0·1017	−1034	−0·0392	−345	−1·7293	101
−8·98	0·5729	−51	−0·2051	−1023	−0·0737	−343	−1·7192	163
−8·96	0·5678	−72	−0·3074	−1010	−0·1080	−338	−1·7029	224
−8·94	0·5606	−92	−0·4084	−994	−0·1418	−333	−1·6805	283
−8·92	0·5514	−111	−0·5078	−973	−0·1751	−328	−1·6522	341
−8·90	0·5403	−131	−0·6051	−949	−0·2079	−319	−1·6181	398
−8·88	0·5272	−149	−0·7000	−922	−0·2398	−311	−1·5783	454
−8·86	0·5123	−167	−0·7922	−893	−0·2709	−302	−1·5329	506
−8·84	0·4956	−185	−0·8815	−859	−0·3011	−291	−1·4823	558
−8·82	0·4771	−202	−0·9674	−823	−0·3302	−279	−1·4265	606
−8·80	0·4569	−218	−1·0497	−784	−0·3581	−267	−1·3659	653
−8·78	0·4351	−233	−1·1281	−743	−0·3848	−253	−1·3006	698
−8·76	0·4118	−247	−1·2024	−699	−0·4101	−239	−1·2308	739
−8·74	0·3871	−262	−1·2723	−653	−0·4340	−224	−1·1569	777
−8·72	0·3609	−273	−1·3376	−606	−0·4564	−208	−1·0792	814
−8·70	0·3336	−285	−1·3982	−555	−0·4772	−191	−0·9978	846
−8·68	0·3051	−296	−1·4537	−503	−0·4963	−174	−0·9132	876
−8·66	0·2755	−306	−1·5040	−451	−0·5137	−156	−0·8256	902
−8·64	0·2449	−314	−1·5491	−395	−0·5293	−138	−0·7354	926
−8·62	0·2135	−321	−1·5886	−340	−0·5431	−119	−0·6428	945
−8·60	0·1814	−327	−1·6226	−284	−0·5550	−100	−0·5483	963
−8·58	0·1487	−333	−1·6510	−226	−0·5650	−81	−0·4520	975
−8·56	0·1154	−336	−1·6736	−169	−0·5731	−61	−0·3545	986
−8·54	0·0818	−340	−1·6905	−111	−0·5792	−41	−0.2559	991
−8·52	0·0478	−341	−1·7016	−52	−0·5833	−21	−0·1568	995
−8·50	0·0137	−341	−1·7068	+5	−0·5854	−2	−0·0573	995
−8·48	0·0204	−341	−1·7063	64	−0·5856	+19	0·0422	990
−8·46	−0·0545	−339	−1·6999	121	−0·5837	38	0·1412	984
−8·44	−0·0884	−335	−1·6878	177	−0·5799	57	0·2396	973
−8·42	−0·1219	−332	−1·6701	233	−0·5742	77	0·3369	960
−8·40	−0·1551	−327	−1·6468	288	−0·5665	96	0·4329	942
−8·38	−0·1878	−320	−1·6180	341	−0·5569	115	0·5271	923
−8·36	−0·2198	−313	−1·5839	393	−0·5454	133	0·6194	900
−8·34	−0·2511	−305	−1·5446	444	−0·5321	151	0·7094	874
−8·32	−0·2816	−295	−1·5002	493	−0·5170	168	0·7968	846
−8·30	−0·3111	−285	−1·4509	539	−0·5002	184	0·8814	814
−8·28	−0·3396	−273	−1·3970	585	−0·4818	200	0·9628	781
−8·26	−0·3669	−262	−1·3385	627	−0·4618	216	1·0409	744
−8·24	−0·3931	−248	−1·2758	667	−0·4402	230	1·1153	706
−8·22	−0·4179	−235	−1·2091	706	−0·4172	244	1·1859	665
−8·20	−0·4414	−220	−1·1385	741	−0·3928	257	1·2524	623
−8·18	−0·4634	−206	−1·0644	775	−0·3671	269	1·3147	578
−8·16	−0·4840	−189	−0·9869	804	−0·3402	280	1·3725	532
−8·14	−0·5029	−173	−0·9065	832	−0·3122	290	1·4257	484
−8·12	−0·5202	−156	−0·8233	857	−0·2832	299	1·4741	435
−8·10	−0·5358	−139	−0·7376	878	−0·2533	307	1·5176	385
−8·08	−0·5497	−121	−0·6498	897	−0·2226	315	1·5561	334

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−8·08	−0·5497	−121	−0·6498	897	−0·2226	315	1·5561	334
−8·06	−0·5618	−103	−0·5601	914	−0·1911	321	1·5895	282
−8·04	−0·5721	−84	−0·4687	925	−0·1590	326	1·6177	229
−8·02	−0·5805	−66	−0·3762	936	−0·1264	330	1·6406	176
−8·00	−0·5871	−47	−0·2826	942	−0·0934	333	1·6582	123
−7·98	−0·5918	−29	−0·1884	946	−0·0601	335	1·6705	69
−7·96	−0·5947	− 9	−0·0938	947	−0·0266	335	1·6774	16
−7·94	−0·5956	+10	0·0009	944	0·0069	336	1·6790	−38
−7·92	−0·5946	28	0·0953	938	0·0405	334	1·6752	−90
−7·90	−0·5918	47	0·1891	931	0·0739	332	1·6662	−143
−7·88	−0·5871	66	0·2822	919	0·1071	329	1·6519	−194
−7·86	−0·5805	84	0·3741	905	0·1400	324	1·6325	−246
−7·84	−0·5721	102	0·4646	888	0·1724	318	1·6079	−295
−7·82	−0·5619	119	0·5534	869	0·2042	313	1·5784	−343
−7·80	−0·5500	137	0·6403	846	0·2355	305	1·5441	−391
−7·78	−0·5363	153	0·7249	822	0·2660	296	1·5050	−436
−7·76	−0·5210	169	0·8071	795	0·2956	288	1·4614	−481
−7·74	−0·5041	185	0·8866	765	0·3244	277	1·4133	−523
−7·72	−0·4856	200	0·9631	733	0·3521	267	1·3610	−564
−7·70	−0·4656	215	1·0364	700	0·3788	255	1·3046	−602
−7·68	−0·4441	228	1·1064	664	0·4043	243	1·2444	−639
−7·66	−0·4213	240	1·1728	626	0·4286	229	1·1805	−673
−7·64	−0·3973	254	1·2354	587	0·4515	216	1·1132	−706
−7·62	−0·3719	264	1·2941	547	0·4731	201	1·0426	−736
−7·60	−0·3455	275	1·3488	503	0·4932	186	0·9690	−763
−7·58	−0·3180	284	1·3991	460	0·5118	171	0·8927	−788
−7·56	−0·2896	294	1·4451	416	0·5289	155	0·8139	−810
−7·54	−0·2602	301	1·4867	369	0·5444	138	0·7329	−831
−7·52	−0·2301	308	1·5236	323	0·5582	121	0·6498	−847
−7·50	−0·1993	314	1·5559	275	0·5703	105	0·5651	−863
−7·48	−0·1679	319	1·5834	227	0·5808	87	0·4788	−874
−7·46	−0·1360	323	1·6061	179	0·5895	69	0·3914	−884
−7·44	−0·1037	326	1·6240	130	0·5964	52	0·3030	−890
−7·42	−0·0711	328	1·6370	81	0·6016	34	0·2140	−894
−7·40	−0·0383	330	1·6451	32	0·6050	16	0·1246	−896
−7·38	−0·0053	329	1·6483	−17	0·6066	−2	0·0350	−894
−7·36	0·0276	329	1·6466	−64	0·6064	−20	−0·0544	−890
−7·34	0·0605	327	1·6402	−113	0·6044	−38	−0·1434	−884
−7·32	0·0932	324	1·6289	−160	0·6006	−55	−0·2318	−874
−7·30	0·1256	321	1·6129	−207	0·5951	−72	−0·3192	−863
−7·28	0·1577	316	1·5922	−252	0·5879	−90	−0·4055	−848
−7·26	0·1893	310	1·5670	−297	0·5789	−106	−0·4903	−832
−7·24	0·2203	304	1·5373	−341	0·5683	−123	−0·5735	−813
−7·22	0·2507	297	1·5032	−383	0·5560	−139	−0·6548	−792
−7·20	0·2804	289	1·4649	−424	0·5421	−154	−0·7340	−769
−7·18	0·3093	280	1·4225	−463	0·5267	−170	−0·8109	−743
−7·16	0·3373	270	1·3762	−502	0·5097	−184	−0·8852	−716

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−7·16	0·3373	270	1·3762	−502	0·5097	−184	−0·8852	−716
−7·14	0·3643	260	1·3260	−538	0·4913	−199	−0·9568	−687
−7·12	0·3903	249	1·2722	−573	0·4714	−211	−1·0255	−655
−7·10	0·4152	237	1·2149	−606	0·4503	−225	−1·0910	−623
−7·08	0·4389	225	1·1543	−637	0·4278	−236	−1·1533	−588
−7·06	0·4614	211	1·0906	−665	0·4042	−248	−1·2121	−553
−7·04	0·4825	198	1·0241	−693	0·3794	−259	−1·2674	−515
−7·02	0·5023	184	0·9548	−717	0·3535	−269	−1·3189	−477
−7·00	0·5207	169	0·8831	−740	0·3266	−277	−1·3666	−437
−6·98	0·5376	154	0·8091	−760	0·2989	−287	−1·4103	−397
−6·96	0·5530	139	0·7331	−779	0·2702	−293	−1·4500	−355
−6·94	0·5669	123	0·6552	−794	0·2409	−300	−1·4855	−313
−6·92	0·5792	107	0·5758	−808	0·2109	−307	−1·5168	−271
−6·90	0·5899	91	0·4950	−820	0·1802	−311	−1·5439	−227
−6·88	0·5990	75	0·4130	−828	0·1491	−315	−1·5666	−183
−6·86	0·6065	57	0·3302	−835	0·1176	−318	−1·5849	−139
−6·84	0·6122	41	0·2467	−839	0·0858	−321	−1·5988	−96
−6·82	0·6163	24	0·1628	−841	0·0537	−322	−1·6084	−51
−6·80	0·6187	8	0·0787	−841	0·0215	−323	−1·6135	−7
−6·78	0·6195	− 10	−0·0054	−839	−0·0108	−323	−1·6142	+37
−6·76	0·6185	− 26	−0·0893	−833	−0·0431	−321	−1·6105	79
−6·74	0·6159	− 43	−0·1726	−826	−0·0752	−320	−1·6026	123
−6·72	0·6116	− 59	−0·2552	−817	−0·1072	−316	−1·5903	165
−6·70	0·6057	− 76	−0·3369	−806	−0·1388	−313	−1·5738	207
−6·68	0·5981	− 91	−0·4175	−792	−0·1701	−308	−1·5531	247
−6·66	0·5890	−107	−0·4967	−776	−0·2009	−303	−1·5284	288
−6·64	0·5783	−123	−0·5743	−759	−0·2312	−297	−1·4996	326
−6·62	0·5660	−137	−0·6502	−740	−0·2609	−289	−1·4670	364
−6·60	0·5523	−152	−0·7242	−718	−0·2898	−283	−1·4306	401
−6·58	0·5371	−166	−0·7960	−695	−0·3181	−273	−1·3905	436
−6·56	0·5205	−180	−0·8655	−670	−0·3454	−265	−1·3469	470
−6·54	0·5025	−193	−0·9325	−644	−0·3719	−255	−1·2999	502
−6·52	0·4832	−206	−0·9969	−616	−0·3974	−245	−1·2497	534
−6·50	0·4626	−217	−1·0585	−586	−0·4219	−233	−1·1963	563
−6·48	0·4409	−229	−1·1171	−556	−0·4452	−233	−1·1400	590
−6·46	0·4180	−240	−1·1727	−524	−0·4675	−210	−1·0810	617
−6·44	0·3940	−250	−1·2251	−490	−0·4885	−197	−1·0193	641
−6·42	0·3690	−260	−1·2741	−457	−0·5082	−185	−0·9552	664
−6·40	0·3430	−268	−1·3198	−421	−0·5267	−171	−0·8888	684
−6·38	0·3162	−276	−1·3619	−386	−0·5438	−157	−0·8204	703
−6·36	0·2886	−284	−1·4005	−348	−0·5595	−143	−0·7501	719
−6·34	0·2602	−290	−1·4353	−311	−0·5738	−128	−0·6782	735
−6·32	0·2312	−296	−1·4664	−274	−0·5866	−113	−0·6047	748
−6·30	0·2016	−301	−1·4938	−234	−0·5979	−99	−0·5299	758
−6·28	0·1715	−306	−1·5172	−196	−0·6078	−83	−0·4541	768
−6·26	0·1409	−309	−1·5368	−157	−0·6161	−68	−0·3773	774
−6·24	0·1100	−311	−1·5525	−118	−0·6229	−52	−0·2999	780

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−6·24	0·1100	−311	−1·5525	−118	−0·6229	−52	−0·2999	780
−6·22	0·0789	−314	−1·5643	−78	−0·6281	−36	−0·2219	782
−6·20	0·0475	−315	−1·5721	−40	−0·6317	−21	−0·1437	784
−6·18	0·0160	−315	−1·5761	0	−0·6338	−6	−0·0653	782
−6·16	−0·0155	−315	−1·5761	39	−0·6344	11	0·0129	780
−6·14	−0·0470	−314	−1·5722	76	−0·6333	26	0·0909	775
−6·12	−0·0784	−312	−1·5646	115	−0·6307	41	0·1684	768
−6·10	−0·1096	−309	−1·5631	153	−0·6266	57	0·2452	760
−6·08	−0·1405	−306	−1·5378	189	−0·6209	72	0·3212	750
−6·06	−0·1711	−301	−1·5189	225	−0·6137	86	0·3962	738
−6·04	−0·2012	−297	−1·4964	261	−0·6051	101	0·4700	723
−6·02	−0·2309	−291	−1·4703	295	−0·5950	116	0·5423	709
−6·00	−0·2600	−285	−1·4408	328	−0·5834	130	0·6132	691
−5·98	−0·2885	−278	−1·4080	362	−0·5704	143	0·6823	673
−5·96	−0·3163	−271	−1·3718	392	−0·5561	156	0·7496	652
−5·94	−0·3434	−262	−1·3326	423	−0·5405	170	0·8148	631
−5·92	−0·3696	−254	−1·2903	452	−0·5235	181	0·8779	608
−5·90	−0·3950	−244	−1·2451	480	−0·5054	194	0·9387	584
−5·88	−0·4194	−234	−1·1971	506	−0·4860	205	0·9971	559
−5·86	−0·4428	−224	−1·1465	531	−0·4655	216	1·0530	532
−5·84	−0·4652	−213	−1·0939	555	−0·4439	226	1·1062	505
−5·82	−0·4865	−202	−1·0379	577	−0·4213	236	1·1567	476
−5·80	−0·5067	−190	−0·9802	598	−0·3977	246	1·2043	446
−5·78	−0·5257	−178	−0·9204	617	−0·3731	254	1·2489	416
−5·76	−0·5435	−166	−0·8587	635	−0·3477	262	1·2905	385
−5·74	−0·5601	−152	−0·7952	651	−0·3215	269	1·3290	353
−5·72	−0·5753	−140	−0·7301	665	−0·2946	276	1·3643	321
−5·70	−0·5893	−126	−0·6636	678	−0·2670	283	1·3964	288
−5·68	−0·6019	−112	−0·5958	689	−0·2387	287	1·4252	254
−5·66	−0·6131	−98	−0·5269	698	−0·2100	293	1·4506	221
−5·64	−0·6229	−85	−0·4571	706	−0·1807	296	1·4727	187
−5·62	−0·6314	−70	−0·3865	713	−0·1511	300	1·4914	152
−5·60	−0·6384	−56	−0·3152	717	−0·1211	302	1·5066	119
−5·58	−0·6440	−41	−0·2435	720	−0·0909	305	1·5185	84
−5·56	−0·6481	−27	−0·1715	721	−0·0604	306	1·5269	50
−5·54	−0·6508	−13	−0·0994	720	−0·0298	307	1·5319	16
−5·52	−0·6521	2	−0·0274	719	0·0009	306	1·5335	−17
−5·50	−0·6519	16	0·0445	715	0·0315	306	1·5318	−52
−5·48	−0·6503	30	0·1160	710	0·0621	305	1·5266	−84
−5·46	−0·6473	44	0·1870	703	0·0926	302	1·5182	−118
−5·44	−0·6429	59	0·2573	695	0·1228	300	1·5064	−150
−5·42	−0·6370	72	0·3268	686	0·1528	296	1·4914	−181
−5·40	−0·6298	86	0·3954	674	0·1824	293	1·4733	−212
−5·38	−0·6212	99	0·4628	662	0·2117	288	1·4521	−243
−5·36	−0·6113	113	0·5290	648	0·2405	283	1·4278	−273
−5·34	−0·6000	125	0·5938	634	0·2688	277	1·4005	−301
−5·32	−0·5875	137	0·6572	616	0·2965	271	1·3704	−329

(Contd.)

t	u	Δu	u′	Δu′	v	Δv	v′	Δv′
−5·32	−0·5875	137	0·6572	616	0·2965	271	1·3704	−329
−5·30	−0 5738	150	0·7188	600	0·3236	264	1·3375	−357
−5·28	−0·5588	162	0·7788	580	0·3500	256	1·3018	−382
−5·26	−0·5426	173	0·8368	561	0·3756	249	1·2636	−408
−5·24	−0·5253	184	0·8929	540	0·4005	241	1·2228	−431
−5·22	−0·5069	194	0·9469	518	0·4246	231	1·1797	−455
−5·20	−0·4875	205	0·9987	496	0·4477	222	1·1342	−476
−5·18	−0·4670	214	1·0483	471	0·4699	212	1·0866	−497
−5·16	−0·4456	224	1·0954	448	0·4911	203	1·0369	−517
−5·14	−0·4232	232	1·1402	422	0·5114	191	0·9852	−534
−5·12	−0·4000	241	1·1824	397	0·5305	181	0·9318	−552
−5·10	−0·3759	248	1·2221	370	0·5486	170	0·8766	−567
−5·08	−0·3511	255	1·2591	343	0·5656	158	0·8199	−582
−5·06	−0·3256	262	1·2934	316	0·5814	147	0·7617	−594
−5·04	−0·2994	268	1·3250	287	0·5961	134	0·7023	−607
−5·02	−0·2726	273	1·3537	260	0·6095	122	0·6416	−617
−5·00	−0·2453	279	1·3797	231	0 6217	110	0·5799	−626
−4·98	−0·2174	282	1·4028	202	0·6327	97	0·5173	−634
−4·96	−0·1892	287	1·4230	173	0·6424	84	0·4539	−640
−4·94	−0·1605	289	1·4403	144	0·6508	72	0·3899	−645
−4·92	−0·1316	292	1·4547	115	0·6580	58	0·3254	−649
−4·90	−0·1024	295	1·4662	86	0·6638	46	0·2605	−652
−4·88	−0·0729	295	1·4748	56	0·6684	33	0·1953	−653
−4·86	−0·0434	297	1·4804	28	0·6717	19	0·1300	−652
−4·84	−0·0137	296	1·4832	−1	0·6736	7	0·0648	−651
−4·82	0·0159	297	1·4831	−29	0·6743	−7	−0·0003	−649
−4·80	0·0456	295	1·4802	−58	0·6736	−20	−0·0652	−644
−4·78	0·0751	294	1·4744	−86	0·6716	−32	−0·1296	−639
−4·76	0·1045	292	1·4658	−113	0·6684	−45	−0·1935	−633
−4·74	0·1337	290	1·4545	−140	0·6639	−58	−0·2568	−626
−4·72	0·1627	286	1·4405	−167	0·6581	−70	−0·3194	−617
−4·70	0·1913	283	1·4238	−193	0·6511	−82	−0·3811	−607
−4·68	0·2196	279	1·4045	−218	0·6429	−94	−0·4418	−596
−4·66	0·2475	274	1·3827	−243	0·6335	−106	−0·5014	−584
−4·64	0·2749	269	1·3584	−267	0·6229	−118	−0·5598	−572
−4·62	0·3018	264	1·3317	−290	0·6111	−129	−0·6170	−557
−4·60	0·3282	257	1·3027	−314	0·5982	−140	−0·6727	−543
−4·58	0·3539	251	1·2713	−335	0·5842	−151	−0·7270	−527
−4·56	0·3790	244	1·2378	−356	0·5691	−161	−0·7797	−511
−4·54	0·4034	237	1·2022	−376	0·5530	−171	−0·8308	−493
−4·52	0·4271	229	1·1646	−395	0·5359	−181	−0·8801	−475
−4·50	0·4500	221	1·1251	−414	0·5178	−190	−0·9276	−457
−4·48	0·4721	212	1·0837	−432	0·4988	−199	−0·9733	−437
−4·46	0·4933	204	1·0405	−448	0·4789	−208	−1·0170	−417
−4·44	0·5137	194	0·9957	−464	0·4581	−215	−1·0587	−397
−4·42	0·5331	185	0·9493	−478	0·4366	−224	−1·0984	−375
−4·40	0·5516	176	0·9015	−493	0·4142	−230	−1·1359	−353

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−4·40	0·5516	176	0·9015	−493	0·4142	−230	−1·1359	−353
−4·38	0·5692	165	0·8522	−504	0·3912	−238	−1·1712	−332
−4·36	0·5857	155	0·8018	−517	0·3674	−244	−1·2044	−309
−4·34	0·6012	145	0·7501	−527	0·3430	−250	−1·2353	−286
−4·32	0·6157	134	0·6974	−536	0·3180	−255	−1·2639	−263
−4·30	0·6291	124	0·6438	−546	0·2925	−261	−1·2902	−240
−4·28	0·6415	112	0·5892	−552	0·2664	−265	−1·3142	−216
−4·26	0·6527	101	0·5340	−559	0·2399	−269	−1·3358	−193
−4·24	0·6628	90	0·4781	−565	0·2130	−273	−1·3551	−169
−4·22	0·6718	79	0·4216	−569	0·1857	−276	−1·3720	−144
−4·20	0·6797	67	0·3647	−573	0·1581	−278	−1·3864	−121
−4·18	0·6864	56	0·3074	−574	0·1303	−281	−1·3985	−97
−4·16	0·6920	44	0·2500	−577	0·1022	−282	−1·4082	−73
−4·14	0·6964	33	0·1923	−576	0·0740	−284	−1·4155	−50
−4·12	0·6997	21	0·1347	−576	0·0456	−284	−1·4205	−26
−4·10	0·7018	10	0·0771	−575	0·0172	−285	−1·4231	−2
−4·08	0·7028	−2	0·0196	−572	−0·0113	−285	−1·4233	21
−4·06	0·7026	−13	−0·0376	−569	−0·0398	−284	−1·4212	43
−4·04	0·7013	−25	−0·0945	−564	−0·0682	−282	−1·4169	67
−4·02	0·6988	−36	−0·1509	−559	−0·0964	−281	−1·4102	88
−4·00	0·6952	−47	−0·2068	−553	−0·1245	−280	−1·4014	111
−3·98	0·6905	−58	−0·2621	−546	−0·1525	−276	−1·3903	132
−3·96	0·6847	−68	−0·3167	−538	−0·1801	−274	−1·3771	153
−3·94	0·6779	−80	−0·3705	−530	−0·2075	−271	−1·3618	174
−3·92	0·6699	−90	−0·4235	−520	−0·2346	−267	−1·3444	194
−3·90	0·6609	−100	−0·4755	−511	−0·2613	−263	−1·3250	213
−3·88	0·6509	−110	−0·5266	−499	−0·2876	−258	−1·3037	233
−3·86	0·6399	−120	−0·5765	−489	−0·3134	−254	−1·2804	251
−3·84	0·6279	−130	−0·6254	−476	−0·3388	−248	−1·2553	269
−3·82	0·6149	−140	−0·6730	−463	−0·3636	−243	−1·2284	287
−3·80	0·6009	−148	−0·7193	−450	−0·3879	−237	−1·1997	303
−3·78	0·5861	−157	−0·7643	−436	−0·4116	−231	−1·1694	319
−3·76	0·5704	−166	−0·8079	−422	−0·4347	−224	−1·1375	334
−3·74	0·5538	−174	−0·8501	−406	−0·4571	−217	−1·1041	349
−3·72	0·5364	−182	−0·8907	−392	−0·4788	−211	−1·0692	363
−3·70	0·5182	−190	−0·9299	−375	−0·4999	−202	−1·0329	377
−3·68	0·4992	−197	−0·9674	−359	−0·5201	−196	−0·9952	389
−3·66	0·4795	−204	−1·0033	−343	−0·5397	−187	−0·9563	401
−3·64	0·4591	−211	−1·0376	−326	−0·5584	−179	−0·9162	412
−3·62	0·4380	−217	−1·0702	−308	−0·5763	−171	−0·8750	422
−3·60	0·4163	−223	−1·1010	−291	−0·5934	−162	−0·8328	432
−3·58	0·3940	−229	−1·1301	−273	−0·6096	−154	−0·7896	441
−3·56	0·3711	−234	−1·1574	−255	−0·6250	−144	−0·7455	448
−3·54	0·3477	−239	−1·1829	−237	−0·6394	−136	−0·7007	457
−3·52	0·3238	−244	−1·2066	−219	−0·6530	−126	−0·6550	463
−3·50	0·2994	−247	−1·2285	−201	−0·6656	−117	−0·6087	468
−3·48	0·2747	−252	−1·2486	−181	−0·6773	−108	−0·5619	474

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−3·48	0·2747	−252	−1·2486	−181	−0·6773	−108	−0·5619	474
−3·46	0·2495	−255	−1·2667	−164	−0·6881	−98	−0·5145	478
−3·44	0·2240	−258	−1·2831	−145	−0·6979	−89	−0·4667	482
−3·42	0·1982	−261	−1·2976	−126	−0·7068	−78	−0·4185	485
−3·40	0·1721	−263	−1·3102	−108	−0·7146	−70	−0·3700	487
−3·38	0·1458	−265	−1·3210	−89	−0·7216	−59	−0·3213	488
−3·36	0·1193	−267	−1·3299	−71	−0·7275	−50	−0·2725	489
−3·34	0·0926	−268	−1·3370	−53	−0·7325	−39	−0·2236	490
−3·32	0·0658	−269	−1·3423	−35	−0·7364	−30	−0·1746	488
−3·30	0·0389	−269	−1·3458	−16	−0·7394	−21	−0·1258	487
−3·28	0·0120	−269	−1·3474	1	−0·7415	−10	−0·0771	486
−3·26	−0·0149	−270	−1·3473	18	−0·7425	−1	−0·0285	483
−3·24	−0·0419	−269	−1·3455	36	−0·7426	9	0·0198	479
−3·22	−0·0688	−267	−1·3419	52	−0·7417	18	0·0677	476
−3·20	−0·0955	−267	−1·3367	70	−0·7399	28	0·1153	471
−3·18	−0·1222	−265	−1·3297	86	−0·7371	37	0·1624	466
−3·16	−0·1487	−263	−1·3211	102	−0·7334	46	0·2090	461
−3·14	−0·1750	−261	−1·3109	118	−0·7288	56	0·2551	454
−3·12	−0·2011	−259	−1·2991	133	−0·7232	65	0·3005	448
−3·10	−0·2270	−256	−1·2858	148	−0·7167	73	0·3453	441
−3·08	−0·2526	−252	−1·2710	163	−0·7094	82	0·3894	433
−3·06	−0·2778	−250	−1·2547	177	−0·7012	91	0·4327	425
−3·04	−0·3028	−245	−1·2370	191	−0·6921	99	0·4752	416
−3·02	−0·3273	−242	−1·2179	204	−0·6822	108	0·5168	408
−3·00	−0·3515	−237	−1·1975	217	−0·6714	115	0·5576	398
−2·98	−0·3752	−233	−1·1758	230	−0·6599	124	0·5974	388
−2·96	−0·3985	−228	−1·1528	242	−0·6475	131	0·6362	379
−2·94	−0·4213	−223	−1·1286	253	−0·6344	138	0·6741	367
−2·92	−0·4436	−218	−1·1033	265	−0·6206	146	0·7108	357
−2·90	−0·4654	−213	−1·0768	275	−0·6060	153	0·7465	346
−2·88	−0·4867	−207	−1·0493	286	−0·5907	159	0·7811	335
−2·86	−0·5074	−201	−1·0207	294	−0·5748	166	0·8146	323
−2·84	−0·5275	−195	−0·9913	305	−0·5582	173	0·8469	311
−2·82	−0·5470	−189	−0·9608	312	−0·5409	179	0·8780	299
−2·80	−0·5659	−183	−0·9296	321	−0·5230	184	0·9079	286
−2·78	−0·5842	−176	−0·8975	329	−0·5046	190	0·9365	275
−2·76	−0·6018	−170	−0·8646	336	−0·4856	195	0·9640	262
−2·74	−0·6188	−163	−0·8310	342	−0·4661	201	0·9902	249
−2·72	−0·6351	−156	−0·7968	348	−0·4460	205	1·0151	236
−2·70	−0·6507	−148	−0·7620	354	−0·4255	210	1·0387	223
−2·68	−0·6655	−142	−0·7266	360	−0·4045	215	1·0610	210
−2·66	−0·6797	−135	−0·6906	363	−0·3830	218	1·0820	198
−2·64	−0·6932	−127	−0·6543	368	−0·3612	222	1·1018	184
−2·62	−0·7059	−120	−0·6175	372	−0·3390	226	1·1202	171
−2·60	−0·7179	−112	−0·5803	375	−0·3164	229	1·1373	158
−2·58	−0·7291	−105	−0·5428	377	−0·2935	232	1·1531	145
−2·56	−0·7396	−97	−0·5051	380	−0·2703	235	1·1676	132

(*Contd.*)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−2·56	−0·7396	−97	−0·5051	380	−0·2703	235	1·1676	132
−2·54	−0·7493	−90	−0·4671	381	−0·2468	237	1·1808	118
−2·52	−0·7583	−81	−0·4290	383	−0.2231	240	1·1926	106
−2·50	−0·7664	−75	−0·3907	384	−01991	242	1·2032	94
−2·48	−0·7739	−66	−0·3523	384	−0·1749	243	1·2126	80
−2·46	−0·7805	−59	−0·3139	384	−0·1506	245	1·2206	68
−2·44	−0·7864	−52	−0·2755	383	−0·1261	246	1·2274	55
−2·42	−0·7916	−43	−0·2372	383	−0·1015	247	1·2329	43
−2·40	−0·7959	−36	−0·1989	381	−0·0768	248	1·2372	31
−2·38	−0·7995	−29	−0·1608	380	−0·0520	248	1·2403	19
−2·36	−0·8024	−20	−0·1228	377	−0·0272	248	1·2422	6
−2·34	−0·8044	−14	−0·0851	376	−0·0024	249	1·2428	−4
−2·32	−0·8058	−5	−0·0475	372	0·0225	248	1·2424	−16
−2·30	−0·8063	1	−0·0103	369	0·0473	248	1·2408	−28
−2·28	−0·8062	9	0·0266	366	0·0721	248	1·2380	−38
−2·26	−0·8053	16	0·0632	362	0·0969	246	1·2342	−49
−2·24	−0·8037	24	0·0994	358	0·1215	245	1·2293	−60
−2·22	−0·8013	31	0·1352	354	0·1460	244	1·2233	−70
−2·20	−0·7982	37	0·1706	348	0·1704	243	1·2163	−80
−2·18	−0·7945	45	0·2054	344	0·1947	240	1·2083	−89
−2·16	−0·7900	51	0·2398	339	0·2187	239	1·1994	−99
−2·14	−0·7849	58	0·2737	333	0·2426	237	1·1895	−09
−2·12	−0·7791	65	0·3070	328	0·2663	235	1·1786	−117
−2·10	−0·7726	71	0·3398	321	0·2898	232	1·1669	−126
−2·08	−0·7655	78	0·3719	315	0·3130	229	1·1543	−134
−2·06	−0·7577	83	0·4034	309	0·3359	227	1·1409	−143
−2·04	−0·7494	90	0·4343	303	0·3586	224	1·1266	−150
−2·02	−0·7404	96	0·4646	296	0·3810	221	1·1116	−158
−2·00	−0·7308	102	0·4942	288	0·4031	217	1·0958	−164
−1·98	−0·7206	107	0·5230	282	0·4248	214	1·0794	−172
−1·96	−0·7099	113	0·5512	275	0·4462	211	1·0622	−178
−1·94	−0·6986	119	0·5787	267	0·4673	207	1·0444	−184
−1·92	−0·6867	123	0·6054	260	0·4880	203	1·0260	−191
−1·90	−0·6744	129	0·6314	253	0·5083	200	1·0069	−196
−1·88	−0·6615	134	0·6567	245	0·5283	195	0·9873	−201
−1·86	−0·6481	139	0·6812	237	0·5478	192	0·9672	−206
−1·84	−0·6342	143	0·7049	229	0·5670	187	0·9466	−211
−1·82	−0·6199	148	0·7278	222	0·5857	183	0·9255	−216
−1·80	−0·6051	152	0·7500	214	0·6040	178	0·9039	−219
−1·78	−0·5899	156	0·7714	206	0·6218	175	0·8820	−229
−1·76	−0.5743	161	0·7920	198	0·6393	169	0·8591	−221
−1·74	−0·5582	164	0·8118	191	0·6562	165	0·8370	−230
−1·72	−0·5418	168	0·8309	182	0·6727	161	0·8140	−233
−1·70	−0·5250	172	0·8491	175	0·6888	156	0·7907	−235
−1·68	−0·5078	175	0·8666	166	0·7044	151	0·7672	−238
−1·66	−0·4903	178	0·8832	159	0·7195	146	0·7434	−240
−1·64	−0·4725	181	0·8991	151	0·7341	142	0·7194	−241

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−1·64	−0·4725	181	0·8991	151	0·7341	142	0·7194	−241
−1·62	−0·4544	184	0·9142	144	0·7483	136	0·6953	−244
−1·60	−0·4360	187	0·9286	135	0·7619	132	0·6709	−244
−1·58	−0·4173	190	0·9421	128	0·7751	127	0·6465	−245
−1·56	−0·3983	192	0·9549	121	0·7878	122	0·6220	−246
−1·54	−0·3791	195	0·9670	113	0·8000	117	0·5974	−247
−1·52	−0·3596	197	0·9783	106	0·8117	112	0·5727	−247
−1·50	−0·3399	198	0·9889	98	0·8229	107	0·5480	−247
−1·48	−0·3201	201	0·9987	91	0·8336	102	0·5233	−246
−1·46	−0·3000	203	1·0078	84	0·8438	97	0·4987	−246
−1·44	−0·2797	204	1·0162	77	0·8535	93	0·4741	−246
−1·42	−0·2593	205	1·0239	71	0·8628	87	0·4495	−244
−1·40	−0·2388	207	1·0310	63	0·8715	83	0·4251	−244
−1·38	−0·2181	208	1·0373	57	0·8798	77	0·4007	−242
−1·36	−0·1973	209	1·0430	50	0·8875	73	0·3765	−241
−1·34	−0·1764	210	1·0480	45	0·8948	68	0·3524	−238
−1·32	−0·1554	211	1·0525	38	0·9016	64	0·3286	−238
−1·30	−0·1343	212	1·0563	32	0·9080	58	0·3048	−235
−1·28	−0·1131	212	1·0595	26	0·9138	54	0·2813	−232
−1·26	−0·0919	212	1·0621	20	0·9192	50	0·2581	−231
−1·24	−0·0707	213	1·0641	15	0·9242	44	0·2350	−228
−1·22	−0·0494	214	1·0656	9	0·9286	41	0·2122	−225
−1·20	−0·0280	213	1·0665	4	0·9327	35	0·1897	−222
−1·18	−0·0067	213	1·0669	−1	0·9362	32	0·1675	−220
−1·16	0·0146	214	1·0668	−6	0·9394	27	0·1455	−216
−1·14	0·0360	213	1·0662	−10	0·9421	22	0·1239	−213
−1·12	0·0573	213	1·0652	−15	0·9443	19	0·1026	−210
−1·10	0·0786	212	1·0637	−20	0·9462	14	0·0816	−207
−1·08	0·0998	212	1·0617	−23	0·9476	10	0·0609	−202
−1·06	0·1210	212	1·0594	−28	0·9486	6	0·0407	−200
−1·04	0·1422	211	1·0566	−31	0·9492	2	0·0207	−195
−1·02	0·1633	210	1·0535	−35	0·9494	−1	0·0012	−192
−1·00	0·1843	210	1·0500	−39	0·9493	−6	−0·0180	−188
−0·98	0·2053	209	1·0461	−42	0·9487	−9	−0·0368	−184
−0·96	0·2262	208	1·0419	−45	0·9478	−13	−0·0552	−180
−0·94	0·2470	207	1·0374	−48	0·9465	−16	−0·0732	−176
−1·92	0·2677	206	1·0326	−50	0·9449	−20	−0·0908	−172
−0·90	0·2883	205	1·0276	−53	0·9429	−24	−0·1080	−167
−0·88	0·3088	204	1·0223	−56	0·9405	−26	−0·1247	−164
−0·86	0·3292	202	1·0167	−57	0·9379	−30	−0·1411	−159
−0·84	0·3494	202	1·0110	−60	0·9349	−33	−0·1570	−155
−0·82	0·3696	200	1·0050	−62	0·9316	−36	−0·1725	−150
−0·80	0·3896	199	0·9988	−63	0·9280	−39	−0·1875	−147
−0·78	0·4095	198	0·9925	−64	0·9241	−42	−0·2022	−142
−0·76	0·4293	197	0·9861	−66	0·9199	−44	−0·2164	−137
−0·74	0·4490	195	0·9795	−67	0·9155	−48	−0·2301	−134
−0·72	0·4685	194	0·9728	−68	0·9107	−50	−0·2435	−129

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
−0·72	0·4685	194	0·9728	−68	0·9107	−50	−0·2435	−129
−0·70	0·4879	193	0·9660	−69	0·9057	−52	−0·2564	−124
−0·68	0·5072	191	0·9591	−69	0·9005	−55	−0.2688	−121
−0·66	0·5263	189	0·9522	−70	0·8950	−58	−0·2809	−116
−0·64	0·5452	189	0·9452	−70	0·8892	−59	−0·2925	−111
−0·62	0·5641	187	0·9382	−70	0·8833	−62	−0·3036	−108
−0·60	0·5828	185	0·9312	−69	0·8771	−64	−0·3144	−103
−0·58	0·6013	184	0·9243	−70	0·8707	−66	−0·3247	−99
−0·56	0·6197	183	0·9173	−69	0·8641	−68	−0·3346	−94
−0·54	0·6380	182	0·9104	−69	0·8573	−69	−0·3440	−91
−0·52	0·6562	180	0·9035	−68	0·8504	−72	−0·3531	−86
−0·50	0·6742	178	0·8967	−66	0·8432	−73	−0·3617	−83
−0·48	0·6920	178	0·8901	−66	0·8359	−75	−0·3700	−78
−0·46	0·7098	176	0·8835	−65	0·8284	−76	−0·3778	−74
−0·44	0·7274	174	0·8770	−63	0·8208	−78	−0·3852	−70
−0·42	0·7448	174	0·8707	−62	0·8130	−79	−0·3922	−67
−0·40	0·7622	172	0·8645	−60	0·8051	−80	−0·3989	−62
−0·38	0·7794	171	0·8585	−59	0·7971	−82	−0·4051	−59
−0·36	0·7965	170	0·8526	−56	0·7889	−83	−0·4110	−55
−0·34	0·8135	169	0·8470	−54	0·7806	−84	−0·4165	−51
−0·32	0·8304	168	0·8416	−52	0·7722	−84	−0·4216	−48
−0·30	0·8472	167	0·8364	−50	0·7638	−86	−0·4264	−44
−0·28	0·8639	166	0·8314	−47	0·7552	−87	−0·4308	−40
−0·26	0·8805	164	0·8267	−44	0·7465	−87	−0·4348	−37
−0·24	0·8969	164	0·8223	−42	0·7378	−88	−0·4385	−34
−0·22	0·9133	164	0·8181	−39	0·7290	−89	−0·4419	−30
−0·20	0·9297	162	0·8142	−35	0·7201	−89	−0·4449	−28
−0·18	0·9459	162	0·8107	−33	0·7112	−90	−0·4477	−24
−0·16	0·9621	161	0·8074	−29	0·7022	−90	−0·4501	−21
−0·14	0·9782	161	0·8045	−25	0·6932	−91	−0·4522	−18
−0·12	0·9943	160	0·8020	−22	0·6841	−91	−0·4540	−14
−0·10	1·0103	160	0·7998	−19	0·6750	−91	−0·4554	−13
−0·08	1·0263	159	0·7979	−14	0·6659	−91	−0·4567	−9
−0·06	1·0422	159	0·7965	−11	0·6568	−92	−0·4576	−6
−0·04	1·0581	159	0·7954	−6	0·6476	−92	−0·4582	−4
−0·02	1·0740	159	0·7948	−2	0·6384	−91	−0·4586	−1
0·00	1·0899	159	0·7946	2	0·6293	−92	−0·4587	1
0·02	1·1058	159	0·7948	7	0·6201	−92	−0·4586	3
0·04	1·1217	159	0·7955	11	0·6109	−91	−0·4583	6
0·06	1·1376	160	0·7966	16	0·6018	−92	−0·4577	9
0·08	1·1536	160	0·7982	21	0·5926	−91	−0·4568	10
0·10	1·1696	160	0·8003	26	0·5835	−91	−0·4558	13
0·12	1·1856	161	0·8029	31	0·5744	−91	−0·4545	15
0·14	1·2017	162	0·8060	36	0·5653	−90	−0·4530	17
0·16	1·2179	162	0·8096	42	0·5563	−90	−0·4513	19
0·18	1·2341	163	0·8138	47	0·5473	−90	−0·4494	20
0·20	1·2504	164	0·8185	53	0·5383	−89	−0·4474	23

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
0·20	1·2504	164	0·8185	53	0·5383	−89	−0·4474	23
0·22	1·2668	166	0·8238	59	0·5294	−89	−0·4451	24
0·24	1·2834	166	0·8297	64	0·5205	−88	−0·4427	26
0·26	1·3000	168	0·8361	71	0·5117	−88	−0·4401	27
0·28	1·3168	169	0·8432	77	0·5029	−87	−0·4374	29
0·30	1·3337	171	0·8509	83	0·4942	−87	−0·4345	30
0·32	1·3508	173	0·8592	90	0·4855	−86	−0·4315	32
0·34	1·3681	175	0·8682	96	0·4769	−85	−0·4283	33
0·36	1·3856	176	0·8778	103	0·4684	−85	−0·4250	34
0·38	1·4032	179	0·8881	110	0·4599	−84	−0·4216	36
0·40	1·4211	181	0·8991	118	0·4515	−83	−0·4180	37
0·42	1·4392	183	0·9109	124	0·4432	−83	−0·4143	37
0·44	1·4575	186	0·9233	132	0·4349	−81	−0·4106	39
0·46	1·4761	189	0·9365	140	0·4268	−81	−0·4067	40
0·48	1·4950	192	0·9505	147	0·4187	−80	−0·4027	41
0·50	1·5142	194	0·9652	156	0·4107	−80	−0·3986	41
0·52	1·5336	198	0·9808	163	0 4027	−78	−0·3945	42
0·54	1·5534	201	0·9971	172	0·3949	−78	−0·3903	43
0·56	1·5735	205	1·0143	181	0·3871	−76	−0·3860	44
0·58	1·5940	208	1·0324	189	0·3795	−76	−0·3816	44
0·60	1·6148	212	1·0513	199	0·3719	−75	−0·3772	45
0·62	1·6360	217	1·0712	207	0·3644	−74	−0·3727	46
0·64	1·6577	220	1·0919	217	0·3570	−74	−0·3681	46
0·66	1·6797	225	1·1136	227	0·3496	−72	−0·3635	46
0·68	1·7022	230	1·1363	236	0·3424	−71	−0·3589	47
0·70	1·7252	234	1·1599	247	0·3353	−71	−0·3542	47
0·72	1·7486	240	1·1846	257	0·3282	−69	−0·3495	47
0·74	1·7726	244	1·2103	268	0·3213	−68	−0·3448	48
0·76	1·7970	251	1·2371	278	0·3145	−68	−0·3400	48
0·78	1·8221	255	1·2649	290	0·3077	−67	−0·3352	48
0·80	1·8476	262	1·2939	302	0·3010	−65	−d·3304	48
0·82	1·8738	268	1·3241	313	0·2945	−65	−0·3256	48
0·84	1·9006	275	1·3554	325	0·2880	−63	−0·3208	49
0·86	1·9281	0·0280	1·3879	338	0·2817	−63	−0·3159	48
0·88	1·9561	0·0288	1·4217	351	0·2754	−62	−0·3111	49
0·90	1·9849	0·029	1·4568	364	0·2692	−61	−0·3062	48
0·92	2·014	0·031	1·4932	377	0·2631	−59	−0·3014	49
0·94	2·045	31	1·5309	392	0·2572	−59	−0·2965	48
0·96	2·076	31	1·5701	406	0·2513	−58	−0·2917	48
0·98	2·071	33	1·6107	420	0·2455	−57	−0·2869	48
1·00	2·140	34	1·6527	436	0·2398	−56	−0·2821	48
1·02	2·174	34	1·6963	451	0·2342	−55	−0·2773	48
1·04	2·208	35	1·7414	467	0·2287	−54	−0·2725	47
1·06	2·243	37	1·7881	484	0·2233	−53	−0·2678	47
1·08	2·280	37	1·8365	501	0·2180	−52	−0·2631	47
1·10	2·317	38	1·8866	0·0519	0·2128	−51	−0·2584	47
1·12	2·355	39	1·9385	0·0537	0·2077	−0·0051	−0·2537	46

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
1·12	2·355	39	1·9385	0·0537	0·2077	−0·0051	−0·2537	46
1·14	2·394	41	1·9922	0·056	0·2026	−0·0049	−0·2491	46
1·16	2·435	41	2·048	0·057	0·1977	−0·00484	−0·2445	46
1·18	2·476	43	2·105	60	0·19286	−476	−0·2399	45
1·20	2·519	44	2·165	61	0·18810	−466	−0·2354	45
1·22	2·563	45	2·226	64	0·18344	−457	−0·2309	45
1·24	2·608	46	2·290	65	0·17887	−448	−0·2264	44
1·26	2·654	48	2·355	68	0·17439	−440	−0·2220	44
1·28	2·702	49	2·423	71	0·16999	−431	−0·2176	43
1·30	2·751	51	2·494	73	0·16568	−422	−0·2133	43
1·32	2·802	52	2·567	75	0·16146	−414	−0·2090	42
1·34	2·854	54	2·642	78	0·15732	−405	−0·2048	0·0042
1·36	2·908	55	2·720	80	0·15327	−397	−0·2006	0·0042
1·38	2·963	57	2·800	83	0·14930	−389	−0·19643	0·00410
1·40	3·020	58	2·883	86	0·14541	−381	−0·19233	0·00405
1·42	3·078	60	2·969	89	0·14160	−372	−0·18828	399
1·44	3·138	63	3·058	92	0·13788	−365	−0·18429	395
1·46	3·201	63	3·150	95	0·13423	−357	−0·18034	389
1·48	3·264	66	3·245	98	0·13066	−349	−0·17645	384
1·50	3·330	68	3·343	102	0·12717	−341	−0·17261	379
1·52	3·398	70	3·445	105	0·12376	−334	−0·16882	374
1·54	3·468	72	3·550	109	0·12042	−326	−0·16508	368
1·56	3·540	75	3·659	112	0·11716	−320	−0·16140	363
1·58	3·615	76	3·771	116	0·11396	−312	−0·15777	357
1·60	3·691	79	3·887	120	0·11084	−304	−0·15420	352
1·62	3·770	81	4·007	124	0·10780	−298	−0·15068	347
1·64	3·851	84	4·131	129	0·10482	−291	−0·14721	341
1·66	3·935	87	4·260	133	0·10191	−285	−0·14380	336
1·68	4·022	89	4·393	137	0·09906	−277	−0·14044	330
1·70	4·111	92	4·530	142	0·09629	−271	−0·13714	324
1·72	4·203	95	4·672	147	0·09358	−264	−0·13390	319
1·74	4·298	98	4·819	153	0·09094	−259	−0·13071	314
1·76	4·396	101	4·972	157	0·08835	−252	−0·12757	308
1·78	4·497	104	5·129	163	0·08583	−246	−0·12449	303
1·80	4·601	108	5·292	168	0·08337	−240	−0·12146	298
1·82	4·709	111	5·460	175	0·08097	−234	−0·11848	291
1·84	4·820	114	5·635	180	0·07863	−228	−0·11557	287
1·86	4·934	118	5·815	187	0·07635	−223	−0·11270	282
1·88	5·052	122	6·002	193	0·07412	−217	−0·10988	276
1·90	5·174	126	6·195	200	0·07195	−211	−0·10712	271
1·92	5·300	130	6·395	207	0·06984	−206	−0·10441	265
1·94	5·430	134	6·602	215	0·06778	−201	−0·10176	260
1·96	5·564	139	6·817	221	0·06577	−196	−0·09916	256
1·98	5·703	143	7·038	230	0·06381	−191	−0·09660	250
2·00	5·846	147	7·268	238	0·06190	−186	−0·09410	245
2·02	5·993	153	7·506	247	0·06004	−180	−0·09165	240
2·04	6·146	158	7·753	255	0·05824	−177	−0·08925	235

(Contd.)

t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
2·04	6·146	158	7·753	255	0·05824	−177	−0·08925	235
2·06	6·304	162	8 008	264	0·05647	−171	−0·08690	230
2·08	6·466	169	8·272	274	0·05476	−167	−0·08460	226
2·10	6·635	173	8·546	284	0·05309	−162	−0·08234	220
2·12	6·808	180	8·830	293	0·05147	−159	−0·08014	216
2·14	6·988	185	9·123	305	0·04988	−153	−0·07798	211
2·16	7·173	192	9·428	315	0·04835	−150	−0·07587	207
2·18	7·365	198	9·743	327	0·04685	−146	−0·07380	202
2·20	7·563	205	10·070	339	0·04539	−141	−0·07178	197
2·22	7·768	212	10·409	351	0·04398	−138	−0·06981	194
2·24	7·980	218	10·760	364	0·04260	−134	−0·06787	188
2·26	8·198	227	11·124	378	0·04126	−130	−0·06599	185
2·28	8·425	234	11·502	391	0·03996	−126	−0·06414	180
2·30	8·659	241	11·893	405	0·03870	−123	−0·06234	176
2·32	8·900	251	12·298	421	0·03747	−120	−0·06058	171
2·34	9·151	258	12·719	463	0·03627	−116	−0·05887	168
2·36	9·409	268	13·155	452	0·03511	−112	−0·05719	164
2·38	9·677	277	13·607	469	0·03399	−110	−0·05555	160
2·40	9·954	286	14·076	487	0·03289	−106	−0·05395	156
2·42	10·240	296	14·563	505	0·03183	−104	−0·05239	152
2·44	10·536	307	15 068	524	0·03079	−100	−0·05087	148
2·46	10·843	317	15·592	543	0·02979	−97	−0·04939	145
2·48	11·160	328	16·135	564	0·02882	−95	−0·04794	141
2·50	11·488	340	16·699	585	0·02787	−91	−0·04653	138
2·52	11·828	352	17·284	608	0 02696	−89	−0·04515	134
2·54	12·180	364	17·892	630	0·02607	−87	−0·04381	131
2·56	12·544	377	18·522	654	0·02520	−83	−0·04250	127
2·58	12·921	390	19·176	0·680	0·02437	−82	−0·04123	124
2·60	13·311	404	19·856	0·70	0·02355	−78	−0·03999	121
2·62	13·715	419	20·56	73	0·02277	−77	−0·03878	118
2·64	14·134	433	21·29	76	0·02200	−74	−0·03760	114
2·66	14·567	449	22·05	79	0·02126	−0·00072	−0·03646	112
2·68	15·016	465	22·84	82	0·02054	−0·00069	−0·03534	109
2·70	15·481	482	23·66	86	0·019849	−0·000674	−0·03425	105
2·72	15·963	499	24·52	88	0·019175	−0·000654	−0·03320	103
2·74	16·462	517	25·40	92	0·018521	−633	−0·03217	100
2·76	16·979	536	26·32	96	0·017888	−614	−0·03117	98
2·78	17·515	556	27·28	99	0·017274	−594	−0·03019	94
2·80	18·071	575	28·27	1·03	0·016680	−576	−0·02925	93
2·82	18·646	0·597	29·30	1·07	0·016104	−557	−0·02832	89
2·84	19·243	0·619	30·37	1·12	0·015547	−540	−0·02743	87
2·86	19·862	0·64	31·49	1·16	0·015007	−523	−0·02656	85
2·88	20·50	0·67	32·65	1·20	0·014484	−506	−0·02571	82
2·90	21·17	69	33·85	1·25	0·013978	−489	−0·02489	80
2·92	21·86	71	35·10	1·30	0·013489	−474	−0·02409	78
2·94	22·57	74	36·40	1·36	0·013015	−459	−0·02331	75
2·96	23·31	77	37·76	1·40	0·012556	−444	−0·02256	73

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t	u	Δu	u'	$\Delta u'$	v	Δv	v'	$\Delta v'$
2·96	23·31	77	37·76	1·40	0·012556	−444	−0·02256	73
2·98	24·08	80	39·16	1·47	0·012112	−429	−0·02183	71
3·00	24·88	83	40·63	1·52	0·011683	−416	−0·02112	0·00070
3·02	25·71	86	42·15	1·59	0 011267	−401	−0·02042	0·00067
3·04	26·57	89	43·74	1.64	0·010866	−389	−0·019754	0·000651
3·06	27·46	92	45·38	1·72	0·010477	−376	−0·019103	631
3·08	28·38	96	47·10	1·78	0·010101	−363	−0·018472	613
3·10	29·34	1·00	48·88	1·86	0·009738	−351	−0·017859	595
3·12	30·34	1·03	50·74	1·93	0·009387	−340	−0·017264	577
3·14	31·37	1·08	52·67	2·01	0·009047	−328	−0·016687	559
3·16	32·45	1·11	54·68	2·09	0·008719	−317	−0·016128	543
3·18	33·56	1·16	56·77	2·18	0·008402	−306	−0·015585	526
3·20	34·72	1·20	58·95	2·26	0·008096	−296	−0·015059	511
3·22	35·92	1·25	61·21	2·37	0·007800	−286	−0·014548	494
3·24	37·17	1·29	63·58	2·45	0·007514	−277	−0·014054	479
3·26	38·46	1·35	66·03	2·56	0·007237	−266	−0·013575	465
3·28	39·81	1·40	68·59	2·67	0·006971	−258	−0·013110	450
3·30	41·21	1·45	71·26	2·77	0·006713	−249	−0·012660	436
3·32	42·66	1·51	74·03	2·89	0·006464	−240	−0·012224	423
3·34	44·17	1·57	76·92	3·02	0·006224	−232	−0·011801	409
3·36	45·74	1·63	79·94	3·13	0·005992	−224	−0·011392	396
3·38	47·37	1·69	83·07	3·27	0·005768	−216	−0·010996	384
3·40	49·06	1·76	86·34	3·41	0·005552	−209	−0·010612	371
3·42	50·82	1·83	89·75	3·55	0·005343	−201	−0·010241	360
3·44	52·65	1·90	93·30	3·69	0·005142	−194	−0·009881	348
3·46	54·55	1·98	96·99	3·86	0·004948	−187	−0·009533	337
3·48	56·53	2·06	100·85	4·02	0·004761	−181	−0·009196	326
3·50	58·59	2·14	104·87	4·18	0·004580	−174	−0·008870	315
3·52	60·73	2·22	109·05	4·37	0·004406	−168	−0·008555	305
3·54	62·95	2·32	113·42	4·55	0·004238	−162	−0·008250	295
3·56	65·27	2·40	117·97	4·75	0·004076	−156	−0·007955	286
3·58	67·67	2·51	122·72	4·94	0·003920	−151	−0·007669	276
3·60	70·18	2·60	127·66	5·16	0·003769	−145	−0·007393	267
3·62	72·78	2·71	132·82	5·39	0·003626	−140	−0·007126	258
3·64	75·49	2·82	138·21	5·61	0·003484	−135	−0·006868	249
3·66	78·31	2·94	143·82	5·85	0·003349	−130	−0·006619	241
3·68	81·25	3·05	149·67	6·11	0·003219	−125	−0·006378	233
3·70	84·30	3·18	155·78	6·38	0·003094	−121	−0·006145	225
3·72	87·48	3·31	162·16	6·64	0·002973	−116	−0·005920	217
3·74	90·79	3·44	168·80	6·94	0·002857	−112	−0·005703	211
3·76	94·23	3·59	175·74	7·22	0·002745	−108	−0·005492	202
3·78	97·82	3·73	182·96	7·58	0·002637	−103	−0·005290	196
3·80	101·55	3·89	190·54	7·88	0·002534	−100	−0·005094	190
3·82	105·44	4·05	198·42	8·28	0.002434	−97	−0·004904	182
3·84	109·49	4·22	206·7	8·5	0·002337	−92	−0·004722	177
3·86	113·71	4·39	215·2	9·0	0·002245	−90	−0·004545	170

(Contd.)

t	u	Δu	u'	$\Delta u'$	$1000\,v$	$1000\,\Delta v$	$1000\,v'$	$1000\,\Delta v'$
3·86	113·71	4·39	215·2	9·0	2·245	−0·090	−4·545	0·170
3·88	118·10	4·58	224·2	9·4	2·155	−0·085	−4·375	0·164
3·90	122·68	4·77	233·6	9·8	2·070	−0·083	−4·211	159
3·92	127·45	4·97	243·4	10·2	1·9870	−0·0795	−4·092	153
3·94	132·42	5·18	253·6	10·6	1·9075	−0·0765	−3·899	148
3·96	137·60	5·39	264·2	11·2	1·8310	−736	−3·751	142
3·98	142·99	5·63	275·4	11·6	1·7574	−708	−3·609	137
4·00	148·62	5·86	287·0	12·2	1·6866	−681	−3·472	133
4·02	154·48	6·11	299·2	12·7	1·6185	−655	−3·339	128
4·04	160·59	6·36	311·9	13·2	1·5530	−630	−3·211	123
4·06	166·95	6·64	325·1	13·9	1·4900	−605	−3·088	119
4·08	173·59	6·93	339·0	14·5	1·4295	−583	−2·969	114
4·10	180·52	7·22	353·5	15·1	1·3712	−560	−2·855	111
4·12	187·74	7·53	368·6	15·8	1·3152	−538	−2·744	106
4·14	195·27	7·8	384·4	16·5	1·2614	−517	−2·638	103
4·16	203·1	8·2	400·9	17·3	1.2097	−497	−2·535	98
4·18	211·3	8·6	418·2	18·1	1·1600	−478	−2·437	96
4·20	219·9	8·9	436·3	18·9	1·1122	−459	−2·341	91
4·22	228·8	9·3	455·2	19·7	1·0663	−441	−2·250	89
4·24	238·1	9·7	474·9	20·7	1·0222	−424	−2·161	0·085
4·26	247·8	10·1	495·6	21·5	0·9798	−407	−2·076	0·082
4·28	257·9	10·6	517·1	22·6	0·9391	−391	−1·9944	0·0789
4·30	268·5	11·0	539·7	23·6	0·9000	−375	−1·9155	759
4·32	279·5	11·5	563·3	24·7	0·8625	−361	−1·8396	732
4·34	291·0	12·0	588·0	25·9	0·8264	−346	−1·7664	703
4·36	303·0	12·6	613·9	27·0	0·7918	−332	−1·6961	678
4·38	315·6	13·1	640·9	28·3	0·7586	−319	−1·6283	652
4·40	328·7	13·6	669·2	29·6	0·7267	−307	−1·5631	627
4·42	342·3	14·3	698·8	30·9	0·6960	−294	−1·5004	604
4·44	356·6	14·9	729·7	32·4	0·6666	−282	−1·4400	580
4·46	371·5	15·6	762·1	33·9	0·6384	−271	−1·3820	559
4·48	387·1	16·3	796·0	35·5	0·6113	−259	−1·3261	537
4·50	403·4	17·0	831·5	37·2	0·5854	−250	−1·2724	517
4·52	420·4	17·7	868·7	38·9	0·5604	−239	−1·2207	497
4·54	438·1	18·6	907·6	40·7	0·5365	−229	−1·1710	477
4·56	456·7	19·4	948·7	42·6	0·5136	−220	−1·1233	459
4·58	476·1	20·3	990·9	44·6	0·4916	−211	−1·0774	442
4·60	496·4	21·1	1035·5	46·8	0·4705	−203	−1·0332	424
4·62	517·5	22·2	1082·3	48·9	0·4502	−194	−0·9908	408
4·64	539·7	23·1	1131·2	51·3	0·4308	−186	−0·9500	392
4·66	562·8	24·2	1182·5	53·6	0·4122	−178	−0·9108	377
4·68	587·0	25·3	1236·1	56·3	0·3944	−171	−0·8731	362
4·70	612·3	26·4	1292·4	58·9	0·3773	−164	−0·8369	347
4·72	638·7	27·6	1351·3	61·7	0·3609	−157	−0·8022	344
4·74	666·3	28·9	1413·0	64·7	0·3452	−151	−0·7688	311
4·76	695·2	30·2	1477·7	67·7	0·3301	−144	−0·7367	308

(Contd.)

t	u	Δu	u'	$\Delta u'$	$1000\,v$	$1000\,\Delta v$	$1000\,v'$	$1000\,\Delta v'$
4·76	695·2	30·2	1477·7	67·7	0·3301	−144	−0·7367	308
4·78	725·4	31·7	1545·4	71·0	0·3157	−138	−0·7059	296
4·80	757·1	33·0	1616·4	74·4	0·3019	−132	−0·6763	284
4·82	790·1	34·6	1690·8	78·0	0·2887	−127	−0·6479	272
4·84	824·7	36·2	1768·8	81·8	0·2760	−122	−0·6207	262
4·86	860·9	37·9	1850·6	85·6	0·2638	−116	−0·5945	251
4·88	898·8	39·6	1936·2	90·0	0·2522	−112	−0·5694	242
4·90	938·4	41·4	2026	94	0·2410	−106	−0·5452	231
4·92	979·8	43·4	2120	99	0·2304	−102	−0·5221	222
4·94	1023·2	45·4	2219	104	0·2202	− 98	−0·4999	213
4·96	1068·6	47·6	2323	108	0·2104	− 94	−0·4786	205
4·98	1116·2	49·7	2431	114	0·2010	−0·0090	−0·4581	196
5·00	1165·9	52·1	2545	119	0·19204	−0·00858	−0·4385	188
5·02	1218·0	54·5	2664	126	0·18346	−0·00822	−0·4197	180
5·04	1272·5	57·1	2790	131	0·17524	−0·00786	−0 4017	173
5·06	1329·6	59·8	2912	138	0·16738	−752	−0·3844	166
5·08	1389·4	62·6	3059	144	0·15986	−719	−0·3678	159
5·10	1452·0	65·6	3203	152	0·15267	−689	−0·3519	153
5·12	1517·6	68·7	3355	159	0·14578	−658	−0·3366	146
5·14	1586·3	71·9	3514	167	0·13920	−630	−0·3220	140
5·16	1658·2	75·4	3681	176	0·13290	−603	−0·3080	134
5·18	1733·4	78·9	3857	184	0·12687	−576	−0·2946	129
5·20	1812·5	82·8	4041	193	0·12111	−551	−0·2817	123
5·22	1895·3	86·7	4234	203	0·11560	−527	−0·2694	118
5·24	1982·0	91	4437	213	0·11033	−504	−0·2576	113
5·26	2073	95	4650	223	0·10529	−481	−0·2463	109
5·28	2168	100	4873	235	0·10048	−461	−0·2354	104
5·30	2268	104	5108	246	0·09587	−440	−0·2250	99
5·32	2372	110	5354	259	0·09147	−420	−0·2151	0·0095
5·34	2482	115	5613	271	0·08727	−402	−0·2056	0·0092
5·36	2597	120	5884	286	0·08325	−384	−0·19644	0·00873
5·38	2717	127	6170	299	0·07941	−367	−0·18771	836
5·40	2844	132	6469	315	0·07574	−351	−0·17935	801
5·42	2976	139	6784	331	0·07223	−335	−0·17134	766
5·44	3115	146	7115	347	0·06888	−320	−0·16368	735
5·46	3261	153	7462	365	0·06568	−305	−0·15635	702
5·48	3414	160	7827	384	0·06263	−292	−0·14933	671
5·50	3574	169	8211	403	0·05971	−279	−0·14262	643
5·52	3743	176	8614	424	0·05692	−266	−0·13619	614
5·54	3919	185	9038	445	0·05426	−255	−0·13005	588
5·56	4104	194	9483	468	0·05171	−242	−0·12417	563
5·58	4298	204	9951	492	0·04929	−232	−0·11854	538
5·60	4502	214	10443	517	0·04697	−221	−0·11316	514
5·62	4716	225	10960	543	0·04476	−211	−0·10802	492
5·64	4941	236	11503	572	0·04265	−202	−0·10310	471
5·66	5177	247	12075	601	0·04063	−192	−0·09839	450
5·68	5424	260	12676	632	0·03871	−183	−0·09389	430

(Contd.)

t	u	Δu	u'	$\Delta u'$	$1000\,v$	$1000\,\Delta v$	$1000\,v'$	$1000\,\Delta v'$
5·68	5424	260	12676	632	0·03871	−183	−0·09389	430
5·70	5684	273	13308	664	0·03688	−175	−0·08959	411
5·72	5957	286	13972	699	0·03513	−167	−0·08548	393
5·74	6243	301	14671	736	0·03346	−160	−0·08155	375
5·76	6544	316	15407	773	0·03186	−152	−0·07780	359
5·78	6860	331	16180	813	0·03034	−145	−0·07421	343
5·80	7191	349	16993	856	0·02889	−138	−0·07078	327
5·82	7540	365	17849	900	0·02751	−132	−0·06751	314
5·84	7905	385	18749	947	0·02619	−125	−0·06437	299
5·86	8290	404	19696	997	0·02494	−120	−0·06138	285
5·88	8694	424	20693	1049	0·02374	−115	−0·05853	273
5·90	9118	446	21742	1104	0·02259	−109	−0·05580	261
5·92	9564	468	22846	1162	0·02150	−103	−0·05319	248
5·94	10032	492	24008	1223	0·02047	−0·00099	−0·05071	238
5·96	10524	518	25231	1287	0·019475	−0·000944	−0·04833	227
5·98	11042	544	26518	1355	0·018531	−0·000899	−0·04606	216
6·00	11586	571	27873	1427	0·017632	−857	−0·04390	207

t	$10^{-3}u$	$10^{-3}\,\Delta u$	$10^{-3}u'$	$10^{-3}\,\Delta u'$	$10^{6}v$	$10^{6}\,\Delta v$	$10^{6}v'$	$10^{6}\,\Delta v'$
6·00	11·586	0·571	27·87	1·43	17·632	−0·857	−43·90	2·07
6·02	12·157	0·601	29·30	1·50	16·775	−0·817	−41·83	1·98
6·04	12·758	632	30·80	1·58	15·958	−778	−39·85	1·88
6·06	13·390	664	23·38	1·67	15·180	−742	−37·97	1·80
6·08	14·054	698	34·05	1·75	14·438	−706	−36·17	1·71
6·10	14·752	735	35·80	1·85	13·732	−672	−34·46	1·64
6·12	15·487	772	37·65	1·95	13·060	−641	−32·82	1·56
6·14	16·259	812	39·60	2·04	12·419	−610	−31·26	1·49
6·16	17·071	855	41·64	2·16	11·809	−581	−29·77	1·42
6·18	17·926	898	43·80	2·28	11·228	−553	−28·35	1·36
6·20	18·824	946	46·08	2·39	10·675	−527	−26·99	1·29
6·22	19·770	99	48·47	2·53	10·148	−502	−25·70	1·23
6·24	20·76	1·05	51·00	2·66	9·646	−477	−24·47	1·18
6·26	21·81	1·10	53·66	2·80	9·169	−455	−23·29	1·12
6·28	22·91	1·16	56·46	2·96	8·714	−433	−22·17	1·07
6·30	24·07	1·22	59·42	3·11	8·281	−411	−21·10	1·02
6·32	25·29	1·28	62·53	3·28	7·870	−392	−20·08	0·97
6·34	26·57	1·35	65·81	3·46	7·478	−373	−19·113	0·926
6·36	27·92	1·42	69·27	3·65	7·105	−355	−18·187	883
6·38	29·34	1·50	72·92	3·85	6·750	−338	−17·304	840
6·40	30·84	1·58	76·77	4·05	6·412	−321	−16·464	802
6·42	32·42	1·66	80·82	4·27	6·091	−305	−15·662	763

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t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{8}v'$	$10^{8}\Delta v'$
6·42	32·42	1·66	80·82	4·27	6·091	−305	−15·662	763
6·44	34·08	1·76	85·09	4·51	5·786	−291	−14·899	728
6·46	35·82	1·84	89·60	4·76	5·495	−277	−14·171	693
6·48	37·66	1·94	94·36	5·01	5·218	−262	−13·478	660
6·50	39·60	2·04	99·37	5·29	4·956	−251	−12·818	629
6·52	41·64	2·15	104·66	5·57	4·705	−237	−12·189	598
6·54	43·79	2·26	110·23	5·89	4·468	−226	−11·591	571
6·56	46·05	2·38	116·12	6·20	4·242	−215	−11·020	543
6·58	48·43	2·51	122·32	6·55	4·027	−205	−10·477	517
6·60	50·94	2·65	128·87	6·91	3·822	−194	−9·960	492
6·62	53·59	2·79	135·78	7·29	3·628	−184	−9·468	469
6·64	56·38	2·94	143·07	7·69	3·444	−176	−8·999	446
6·66	59·32	3·09	150·76	8·12	3·268	−167	−8·553	425
6·68	62·41	3·26	158·88	8·56	3·101	−158	−8·128	404
6·70	65·67	3·44	167·44	9·04	2·943	−151	−7·724	385
6·72	69·11	3·63	176·48	9·55	2·792	−143	−7·339	366
6·74	72·74	3·82	186·03	10·07	2·649	−136	−6·973	349
6·76	76·56	4·02	196·10	10·6	2·513	−129	−6·624	331
6·78	80·58	4·25	206·7	11·3	2·384	−123	−6·293	315
6·80	84·83	4·48	218·0	11·8	2·261	−116	−5·978	300
6·82	89·31	4·72	229·8	12·5	2·145	−0·111	−5·678	286
6·84	94·03	4·98	242·3	13·3	2·034	−0·105	−5·392	271
6·86	99·01	5·24	255·6	13·9	1·9291	−0·0998	−5·121	258
6·88	104·25	5·54	269·5	14·8	1·8293	−0·0948	−4·863	246
6·90	109·79	5·84	284·3	15·5	1·7345	−899	−4·617	233
6·92	115·63	6·16	299·8	16·5	1·6446	−855	−4·384	222
6·94	121·79	6·50	316·3	17·4	1·5591	−811	−4·162	211
6·96	128·29	6·85	333·7	18·3	1·4780	−770	−3·951	201
6·98	135·14	7·24	352·0	19·4	1·4010	−731	−3·750	191
7·00	142·38	7·63	371·4	20·5	1·3279	−693	−3·559	181
7·02	150·01	8·05	391·9	21·7	1·2586	−658	−3·378	172
7·04	158·06	8·50	413·6	22·9	1·1928	−625	−3·206	164
7·06	166·56	8·97	436·5	24·1	1·1303	−593	−3·042	156
7·08	175·53	9·46	460·6	25·6	1·0710	−562	−2·886	147
7·10	184·99	10·00	486·2	27·0	1·0148	−534	−2·739	141
7·12	194·99	10·5	513·2	28·5	0·9614	−506	−2·598	133
7·14	205·5	11·2	541·7	30·2	0·9108	−480	−2·465	127
7·16	216·7	11·7	571·9	31·9	0·8628	−455	−2·338	121
7·18	228·4	12·4	603·8	33·8	0·8173	−432	−2·217	0·114
7·20	240·8	13·1	637·6	35·6	0·7741	−410	−2·103	0·109
7·22	253·9	13·9	673·2	37·7	0·7331	−389	−1·9944	0·1031
7·24	267·8	14·6	710·9	39·9	0·6942	−368	−1·8913	0·0980
7·26	282·4	15·4	750·8	42·2	0·6574	−349	−1·7933	930
7·28	297·8	16·3	793·0	44·6	0·6225	−331	−1·7003	883
7·30	314·1	17·2	837·6	47·2	0·5894	−314	−1·6120	839
7·32	331·3	18·2	884·8	49·9	0·5580	−298	−1·5281	796

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t	$10^{-3}u$	$10^{-3}\Delta u$	$10^{-3}u'$	$10^{-3}\Delta u'$	$10^{6}v$	$10^{6}\Delta v$	$10^{6}v'$	$10^{6}\Delta v'$
7·32	331·3	18·2	884·8	49·9	0·5580	—298	—1·5281	796
7·34	349·5	19·2	934·7	52·7	0·5282	−282	−1·4485	755
7·36	368·7	20·3	987·4	55·9	0·5000	−267	−1·3730	717
7·38	389·0	21·5	1043·3	59·1	0·4733	−254	−1·3013	681
7·40	410·5	22·7	1102·4	62·4	0·4479	−240	−1·2332	646
7·42	433·2	23·9	1164·8	66·2	0·4239	−227	−1·1686	613
7·44	457·1	25·3	1231·0	70·0	0·4012	−216	−1·1073	581
7·46	482·4	26·8	1301·0	74·0	0·3796	−204	−1·0492	552
7·48	509·2	28·3	1375·0	78·4	0·3592	−194	−0·9940	524
7·50	537·5	29·9	1453·4	82·9	0·3398	−183	−0·9416	496
7 52	567·4	31·5	1536·3	87·8	0·3215	−174	−0·8920	471
7·54	598·9	33·5	1624·1	93·0	0·3041	−164	−0·8449	447
7·56	632·4	35·2	1717·1	98·3	0·2877	−156	−0·8002	423
7·58	667·6	37·4	1815·4	104·2	0·2721	−147	−0·7579	402
7·60	705·0	39·5	1919·6	110	0·2574	−140	−0·7177	381
7·62	744·5	41·7	2030	117	0·2434	−132	−0·6796	361
7·64	786·2	44·2	2147	123	0·2302	−126	−0·6435	343
7·66	830·4	46·7	2270	131	0·2176	−0·0118	−0·6092	324
7·68	877·1	49·1	2401	139	0·2058	−0·0112	−0·5768	308
7·70	926·5	52·3	2540	147	0·19455	−0·01063	−0·5460	292
7·72	978·8	55·2	2687	155	0·18392	−0·01006	−0·5168	276
7·74	1034·0	58·5	2842	165	0·17386	−951	−0·4892	262
7·76	1092·5	61·9	3007	175	0·16435	−901	−0·4630	249
7·78	1154·4	65·5	3182	185	0·15534	−853	−0·4381	235
7·80	1219·9	69·2	3367	195	0·14681	−807	−0·4146	223
7·82	1289·1	73·3	3562	208	0·13874	−763	−0·3923	211
7·84	1362·4	77·6	3770	220	0·13111	−722	−0·3712	200
7·86	1440·0	82·1	3990	233	0·12389	−683	−0·3512	190
7·88	1522·1	86·9	4223	247	0·11706	−647	−0·3322	179
7·90	1609·0	92·0	4470	262	0·11059	−611	−0·3143	171
7·92	1701·0	97·4	4732	277	0·10448	−578	−0·2972	161
7·94	1798·4	103·1	5009	294	0·09870	−547	−0·2811	152
7·96	1901·5	109	5303	312	0·09323	−517	−0·2659	145
7·98	2011	115	5615	330	0·08806	−489	−0·2514	136
8·00	2126	123	5945	351	0·08317	−463	−0·2378	130

t	$10^{-6}u$	$10^{-6}\Delta u$	$10^{-6}u'$	$10^{-6}\Delta u'$	$10^{9}v$	$10^{9}\Delta v$	$10^{9}v'$	$10^{9}\Delta v'$
8·00	2·126	0·123	5·945	0·351	83·17	−4·63	−237·8	13·0
8·02	2·249	0·129	6·296	0·371	78·54	−4·37	−224·8	12·2
8·04	2·378	137	6·667	394	74·17	−4·13	−212·6	11·7
8·06	2·515	146	7·061	418	70·04	−3·91	−200·9	10·9
8·08	2·661	154	7·479	443	66·13	−3·70	−189·96	10·40

(Contd.)

t	$10^{-6}u$	$10^{-6}\,\Delta u$	$10^{-6}u'$	$10^{-6}\,\Delta u'$	$10^{9}v$	$10^{9}\,\Delta v$	$10^{9}v'$	$10^{9}\,\Delta v'$
8·08	2·661	154	7·479	443	66·13	−3·70	−189·96	10·40
8·10	2·815	163	7·922	469	62·43	−3·49	−179·56	9·83
8·12	2·978	173	8·391	498	58·94	−3·30	−169·73	9·32
8·14	3·151	183	8·889	529	55·64	−3·12	−160 41	8·81
8·16	3·334	193	9·418	560	52·52	−2·95	−151·60	8·34
8·18	3·527	206	9·978	594	49·57	−2·78	−143·26	7·89
8·20	3·733	218	10·572	631	46·79	−2·63	−135·37	7·46
8·22	3·951	230	11·203	669	44·16	−2·49	−127·91	7·07
8·24	4·181	245	11·872	710	41·67	−2·35	−120·84	6·67
8·26	4·426	259	12·582	753	39·32	−2·22	−114·17	6·33
8·28	4·685	275	13·335	800	37·10	−2·10	−107·84	5·97
8·30	4·960	291	14·135	848	35·00	−1·98	−101·87	5·65
8·32	5·251	308	14·983	900	33·02	−1·87	−96·22	5·34
8·34	5·559	327	15·883	956	31·15	−1·76	−90·88	5·06
8·36	5·886	347	16·839	1·014	29·39	−1·67	−85·82	4·78
8·38	6·233	368	17·853	1·076	27·72	−1·58	−81·04	4·51
8·40	6·601	390	18·929	1·14	26·14	−1·48	−76·53	4·27
8·42	6·991	413	20·07	1·22	24·66	−1·41	−72·26	4·04
8 44	7·404	439	21·29	1·28	23·25	−1·33	−68·22	3·82
8·46	7·843	465	22·57	1·37	21·92	−1·25	−64·40	3·60
8·48	8·308	493	23·94	1·45	20·67	−1·18	−60·80	3·41
8·50	8·801	523	25·39	1·54	19·492	−1·116	−57·39	3·22
8·52	9·324	555	26·93	1·64	18·376	−1·052	−54·17	3·05
8·54	9·879	589	28·57	1·74	17·324	−0·994	−51·12	2·88
8·56	10·468	624	30·31	1·85	16·330	−0·937	−48·24	2·71
8·58	11·092	663	32·16	1·96	15·393	−0·885	−45·53	2·57
8·60	11·755	703	34·12	2·08	14·508	−0·834	−42·96	2·43
8·62	12·458	746	36·20	2·22	13·674	−0·788	−40·53	2·29
8·64	13·204	791	38·42	2·35	12·886	−0·743	−38·24	2·16
8·66	13·995	840	40·77	2·50	12·143	−0·701	−36·08	2·05
8·68	14·835	892	43·27	2·65	11·442	−0·661	−34·03	1·93
8·70	15·727	947	45·92	2·83	10·781	−0·624	−32·10	1·82
8·72	16·674	1·004	48·75	2·99	10·157	−0·588	−30·28	1·72
8·74	17·678	1·067	51·74	3·19	9·569	−0·554	−28·56	1·63
8·76	18·745	1·132	54·93	3·39	9·015	−0·524	−26·93	1·53
8·78	19·877	1·20	58·32	3·60	8·491	−0·493	−25·40	1·45
8·80	21·08	1·27	61·92	3·82	7·998	−0·465	−23·95	1·37
8·82	22·35	1·36	65·74	4·07	7·533	−0·439	−22·58	1·29
8·84	23·71	1·44	69·81	4·32	7·094	−0·413	−21·29	1·22
8·86	25·15	1·53	74·13	4·60	6·681	−0·390	−20·07	1·15
8·88	26·68	1·62	78·73	4·88	6·291	−0·367	−18·920	1·086
8·90	28·30	1·72	83·61	5·20	5·924	−0·347	−17·834	1·024
8·92	30·02	1·83	88·81	5·52	5·577	−0·326	−16·810	0·966
8·94	31·85	1·95	94·33	5·88	5·251	−0·308	−15·844	0·913
8·96	33·80	2·06	100·21	6·24	4·943	−0·290	−14·931	0·860
8·98	35·86	2·20	106·45	6·65	4·653	−0·273	−14·071	0·812
9·00	38·06		113·10		4·380		−13·259	

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